Bio-economics of a renewable resource in a seasonally varying environment

César Castilho \textsuperscript{a,c}, Pichika D.N. Srinivasu \textsuperscript{b,c,*}

\textsuperscript{a} Departamento de Matemática, Universidade Federal de Pernambuco, Recife PE, CEP 50740-540, Brazil
\textsuperscript{b} Department of Mathematics, Andhra University, Visakhapatnam 530 003, India
\textsuperscript{c} The Abdus Salam ICTP, Strada Costiera 11, Trieste 34100, Italy

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Abstract

In this paper we study the bio-economics of a renewable resource with governing dynamics described by two distinct growth functions (viz., logistic and Gompertz growth functions) in a seasonally varying environment. Seasonality is introduced into the system by taking the involved ecological parameters to be periodic. In this work, we establish a procedure to obtain the optimal path and compute the optimal effort policy which maximizes the net revenue to the harvester for a fairly general optimal control problem and apply this procedure to the considered models to derive some important conclusions. These problems are solved on the infinite horizon. We find that, for both the models, the optimal harvest policy and the corresponding optimal path are periodic after a finite time. We also obtain optimal solution, a suboptimal harvesting policy and the corresponding suboptimal approach path to reach this optimal solution. The key results are illustrated using numerical simulations and we compare the revenues to the harvester along the optimal and suboptimal paths. The general procedure developed in this work, for obtaining the optimal effort policy and the optimal path, has wider applicability.

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* Corresponding author. Tel.: +91 891 2710016; fax: + 91 891 2755324.
E-mail address: pdns@rediffmail.com (P.D.N. Srinivasu).
1. Introduction

There has been considerable study in recent years on optimal management of renewable resources from various perspectives. The need for sustenance of the resources for future generations is the motivating factor for most of the studies in this area. An excellent introduction to optimal management of renewable resources is given by Clark ([1]), wherein he developed optimal harvesting strategies for both single and multiple dimensional deterministic autonomous ecosystems. It is well documented [2–10] that for modelling ecosystems, it is necessary and important to consider models with periodic forcing terms or periodic ecological parameters or both in order to represent the seasonal effects of weather, food supply, mating habits, etc., to which the ecosystems are naturally exposed. Part of the recent work on management and harvesting of renewable resources is driven by the need to derive insight into the management of ecosystems by studying models that mimic nature as closely as possible by introducing periodicity into the models in some form or the other. Hence the ecosystem management studies concentrated more on models that admit periodic solutions and also the models with periodic coefficients.

Recently, Xianning Liu and Lansun Chen [11] studied the global behavior of periodic logistic system with periodic impulsive perturbations. In [12], optimal harvesting policy is obtained for a single population with logistic growth with impulsive harvesting. They study existence and global attractiveness of impulsive periodic solutions for constant and proportionate harvesting and obtain optimal effort policy to maximize the sustainable yield. They compare the continuous harvest policy with the impulsive harvest policy and find that the continuous harvesting policy is superior to the impulsive harvesting policy. Meng Fan and Ke Wang [13] examine exploitation of a single species modelled by time dependent logistic growth with periodic coefficients. Optimal harvest effort that maximizes the annual sustainable yield is obtained for constant as well as periodic harvests. The results obtained in this work generalize the classical results of Clark for a population described by the autonomous logistic equation in renewable resource management. Ling Bai and Ke Wang [14] studied a time dependent logistic equation with periodic coefficients and obtained the optimal harvest effort that maximizes the sustainable yield per unit time. The results in this work generalize the results of Xiaoying Zhang et al. [12]. Maarten de Gee and Johan Grassman [15] developed an algorithm to determine the optimal sustainable yield for seasonally fluctuating biological population. This algorithm is applied to populations with logistic growth and prey – predator systems. There are also studies dealing with optimal management of natural resources under environmental uncertainty ([16,17]). These studies involve stochastic variation in the environment which affect the growth of the renewable resource.

Solving an infinite horizon optimal harvesting problem associated with an autonomous dynamic system, where the harvesting effort enters linearly as a control variable, is to find the singular optimal harvesting policy along with its corresponding singular solution and develop a strategy to reach this singular solution as early as possible using bang–bang controls [1,18,19]. In these studies the singular solution is represented by a constant (an equilibrium solution) and hence the level to which the state has to be driven using the bang–bang controls is known in advance. But, in case of optimal control problems associated with systems involving periodic coefficients or periodic forcing terms with linear control, more often the optimal singular solutions are periodic in nature ([13,15]) and hence the solution of the optimal control problem would be to approach a set, i.e.,
the optimal singular periodic orbit, in a minimum time and remain in that set for all future times. Synthesis of this approach path is an integral part of the optimal solution of the considered problem.

Though the existence of the optimal singular solutions for a few special cases are obtained in the studies made by M. Fan et al. [13] and Marten de Gee et al. [15] among others, the strategy to reach the optimal singular solution from a given initial state of the ecosystem has not been discussed. Implementing the optimal singular control policy from a given initial condition yields only a suboptimal path. For, it is not obvious that the considered dynamic system always initiates on the optimal trajectory. More often, the state from a given initial position has to be driven to the optimal singular state along an optimal path which is termed as optimal approach path. To the authors knowledge, there does not exist a systematic study of constructing the solution for an optimal control problem associated with systems having periodic coefficients or periodic forcing terms with linear control.

In this article, we study bio-economics of a renewable resource with two different growth dynamics involving periodic coefficients to incorporate seasonality in the models. In Section 2, we consider an optimal control problem associated with a fairly general non autonomous equation and characterize its singular solution. This study offers us a technique to construct the optimal singular solution and the associated singular control using a systematic procedure. In Section 3, we study two classical models of population growth with periodic coefficients in presence of harvesting and discuss existence, uniqueness and global stability of a periodic solution for each of the models. In Section 4, we apply the results of Section 2 to optimal harvesting problems associated with the growth models studied in Section 3 and derive associated optimal harvest policies and optimal solutions. We also obtain optimal and suboptimal approach paths to reach these optimal solutions. In fact, the method developed in this article can also be applied to models with periodic forcing terms which admit a unique globally asymptotically stable periodic solution for a given reasonable periodic harvest function, i.e., whenever the harvesting function belongs to the control set of the considered optimal control problem. Novelty lies in deriving the optimal approach paths to reach the optimal solution of which the initial value information is not known explicitly. The method developed in this paper allows us to extract the information about the location of the optimal singular solution’s initial position and the global asymptotic behavior of the dynamic constraint along with the linearity of the hamiltonian in the control variable permits us to construct the most rapid approach path to reach this optimal singular solution. In Section 5, we illustrate the theoretical findings using numerical simulations and highlight the strength of this work followed by conclusions in Section 6.

2. Characterization of a singular solution

In this section we consider an optimal control problem associated with a non autonomous differential equation with a linear control term. Using the Pontryagin’s maximum principle ([20]) we characterize its singular solution and discuss the construction of its singular optimal control. These results are applied to study optimal harvesting problems involving two well known growth models in a later section.
Let us consider the following optimal control problem
\[
\max_{E \in [E_{\text{min}}, E_{\text{max}}]} I(E) = \int_0^{\infty} e^{-\rho t} [p(z,t)Eg(z,t) - c(z,t)E] \, dt
\]
subject to the dynamic equation
\[
\dot{z} = f(z,t) - Eg(z,t),
\]
where \(c(z,t), f(z,t), g(z,t)\) and \(p(z,t)\) are assumed to be \(C^1\) functions. From Pontryagin’s maximum principles, the associated current value Hamiltonian (see [1]) is given by
\[
H = \eta [f(z,t) - Eg(z,t)] + p(z,t)Eg(z,t) - c(z,t)E.
\]
The above Hamiltonian is linear in the control variable \(E\), hence, the optimal solution is a combination of bang–bang and singular controls on the infinite horizon. The dynamics of the associated costate variable \(\eta(t)\) (shadow price for \(z(t)\)) is given by
\[
\dot{\eta} = \rho \eta - \frac{\partial H}{\partial z},
\]
that is
\[
\dot{\eta} = \rho \eta - \eta (f_z - Eg_z) - E (p_z g + pg_z - c_z). \tag{2}
\]
The singular control is characterized by the zero of the switching function along an interval, that is, when
\[
\frac{\partial H}{\partial E} = 0
\]
on an interval. Thus along the singular solution we have
\[
(p - \eta)g - c = 0. \tag{3}
\]
Differentiating (3) we have
\[
\left( \frac{\partial p}{\partial t} + \frac{\partial p}{\partial z} \dot{z} - \dot{\eta} \right) g + (p - \eta) \left( \frac{\partial g}{\partial t} + \frac{\partial g}{\partial z} \dot{z} \right) = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial z} \dot{z}. \tag{4}
\]
Using equations (1)–(3) in (4) we obtain
\[
\frac{\partial}{\partial t} (pg) + f \frac{\partial}{\partial z} (pg) - \left( \rho - \frac{c}{g} \right) \left[ \frac{\partial g}{\partial t} + \frac{\partial g}{\partial z} f + g (p - f_z) \right] = \frac{\partial c}{\partial t} + \frac{\partial c}{\partial z} f. \tag{5}
\]
This is an implicit equation for \(z\) as a function of \(t\) and characterizes the singular solution. (5) with (1) determines the singular control. Clearly, the nature of the singular solution and the singular control depends on the nature of the involved functions \(c(z,t), f(z,t), g(z,t)\) and \(p(z,t)\) along with their partial derivatives with respect to \(t\) and \(z\). Now the optimal control can be constructed by restricting this control to the bounds \([E_{\text{min}}, E_{\text{max}}]\). Hence, for the considered problem, the optimal control is always a combination of bang–bang and singular controls. The singular path will be a singular optimal path if the singular effort policy \(E(t)\) obtained from (5) satisfies \(E(t) \in [E_{\text{min}}, E_{\text{max}}]\) for all \(t\).
3. Two population growth models with harvesting

In this section we investigate the nature of solutions of two population growth models viz., the logistic and Gompertz models [21] modified by a harvesting term in a seasonally varying environment. We assume that the growth rate and carrying capacities of the considered population are periodic in nature of the same period \( T \). Thus, we consider the following models:

\[
\frac{dx}{dt} = x \left\{ a(t) - b(t)x \right\} - E(t)x
\]  \hspace{1cm} (6)

with

\[ x(0) = x_0 > 0, \]

and

\[
\frac{dy}{dt} = y \left\{ \alpha(t) - \beta(t) \ln(y) \right\} - E(t)y
\]  \hspace{1cm} (7)

with

\[ y(0) = y_0 > 0. \]

For biological reasons we will be interested only in non negative solutions of (6) and (7). \( a(t), \alpha(t) \) represent the growth rates of the resource and \( \frac{a(t)}{b(t)} \) and \( e^{\alpha(t)} \) are the carrying capacities of the models (6) and (7), respectively. Throughout the article we will use the notation

\[
\langle f \rangle \equiv \frac{1}{T} \int_0^T f(t) \, dt
\]

for the average of a periodic function \( f \) of period \( T \).

It can be easily verified that the solutions of (6) approach zero asymptotically if \( a(t) - E(t) \) is negative. But the eventual value of this state, \( x(t) \), can not be that easily determined if \( a(t) - E(t) \) happens to be periodic with unrestricted sign. In this case we observe that the eventual state of a solution of (6) very much depends on the average value of the periodic function \( a(t) - E(t) \). The solutions of (6) approach zero eventually if this average value is negative and they approach a nontrivial globally asymptotically stable periodic orbit of period \( T \) if this average value is positive. The following propositions illustrate this behavior.

**Proposition 3.1.** For a periodic harvest effort \( E(t) \) (of period \( T \)), if \( \langle a - E \rangle \) is negative then solutions of (6) eventually approach the zero solution \( x(t) \equiv 0 \).

**Proof.** Eq. (6) is a Ricatti equation and can be linearized doing the substitution \( z = \frac{1}{x} \). The solutions are

\[
x(t) = \frac{x_0 e^{\int_0^t (a(s) - E(s)) \, ds}}{1 + x_0 e^{\int_0^t (a(s) - E(s)) \, ds}}.
\]
By the hypothesis we have

$$\lim_{t \to \infty} \int_{t_0}^{t} (a(s) - E(s)) \, ds = -\infty,$$

and the result follows. □

The next proposition specifies the conditions on the coefficients of (6) and (7) which ensure existence of asymptotically stable periodic solutions for the respective equations.

**Proposition 3.2.** For a periodic harvest effort $E(t)$ (of period $T$), (7) admits a nontrivial globally asymptotically stable periodic orbit of period $T$. Also, if $(a - E)$ is positive then (6) admits a nontrivial globally asymptotically stable periodic orbit of period $T$.

**Proof.** Eq. (7) can be solved doing the change of variables $v = \ln(y)$ and then using Lagrange multipliers. Periodic orbits of period $T$ for (6) and (7) can be found from their general solutions and are given, respectively, by

$$x_p(t) = \frac{e^{T(a - E)} - 1}{\int_{t}^{t+T} b(\tau) e^{\int_{\tau}^{t}(a(s) - E(s)) \, ds} \, d\tau},$$

and

$$y_p(t) = \exp\left\{ \int_{t}^{t+T} (z(\tau) - E(\tau)) e^{\int_{\tau}^{t} h(s) \, ds} \, d\tau \right\}.$$

We first prove global asymptotic stability for $x_p$; consider two non-zero solutions of (6), say $x_1$ and $x_2$. Define the variable $\xi = \frac{1}{x_1} - \frac{1}{x_2}$. Then

$$\frac{d\xi}{dr} = -\frac{1}{x_1} \frac{dx_1}{dr} + \frac{1}{x_2} \frac{dx_2}{dr} = -(a - E) \left( \frac{1}{x_1} - \frac{1}{x_2} \right) = -(a - E) \xi.$$

Therefore

$$\xi(t) = \xi_0 e^{-\int_{t_0}^{t} (a - E) \, dt}.$$

From the hypothesis, we conclude that

$$\lim_{t \to 0} \xi(t) = 0.$$

It follows that any two non-zero solutions of (6) converge asymptotically. Since $x_p$ is a non-trivial solution the result follows.

The proof for $y_p$ is analogous: doing the coordinate change $v = \ln(y)$, (7) gets transformed to

$$\frac{dv}{dr} = z - \beta v.$$

Let $v_1$ and $v_2$ be two non-zero solutions of this equation. Define $\phi = v_1 - v_2$. Then

$$\frac{d\phi}{dr} = -\beta(t) \phi,$$

and as before the result follows. □
4. Optimal harvesting policy in seasonally varying environment

In this section we wish to obtain an optimal harvesting policy $E(t)$ which maximizes the time stream of net revenues $I(E)$, to the harvester, on the infinite horizon when the resource dynamics is governed by either the logistic or the Gompertz equation considered in the previous section. Thus, we have the following optimal control problems:

**Problem 1:**

$$\max_E I(E) = \int_0^\infty e^{-\delta t} [pEx - cE] \, dt$$

subject to

$$\frac{dx}{dt} = x\{a(t) - b(t)x\} - Ex$$

with

$$x(0) = x_0 \quad \text{and} \quad E(t) \in [E_{\min}, E_{\max}]$$

**Problem 2:**

$$\max_E I(E) = \int_0^\infty e^{-\delta t} [pEy - cE] \, dt$$

subject to

$$\frac{dy}{dt} = y\{\alpha(t) - \beta(t) \ln(y)\} - Ey$$

with

$$y(0) = y_0 \quad \text{and} \quad E(t) \in [E_{\min}, E_{\max}]$$

where $\delta$ is the instantaneous annual rate of discount, $p$ is the price per unit harvest, $c$ is the cost per unit effort and $E_{\max}(E_{\min})$ represents the maximum (minimum) allowable effort in the harvesting activity by the harvesting agency. In this work $\delta$, $p$ and $c$ are assumed to be constants.

The function $E(t) \in [E_{\min}, E_{\max}]$ which solves Problem 1 (Problem 2) is the optimal harvest policy and the corresponding solution $x(t)$ of Problem 1 ($y(t)$ of Problem 2) with $E = E(t)$ is the optimal path. We apply the observations made in Section 2 to the problems 1 and 2 and obtain optimal harvesting policies i.e., the values of $E(t)$ such that $I(E)$ is maximized for the respective problems. We obtain the path traced by the solutions $x(t)$ and $y(t)$ with the optimal harvesting policy so that if the populations under study are kept along this path, we are assured of achieving the objective of the harvesting agency.

4.1. Optimal harvest policy for problem 1

Comparing with the optimal control problem discussed in Section 2, we have $p(x, t) = p$, $c(x, t) = c$, $g(x, t) = x$ and $f(x, t) = x(a(t) - b(t)x)$. The singular trajectory of the population is giv-
en by equation (5) which is nothing but the positive solution, denoted by \(x_*(t), t \geq 0\), of the quadratic equation

\[
2b(t)x^2 - x(b(t)c + p(a(t) - \rho)) - \rho c = 0.
\]  
(14)

Using the Eq. (6) we obtain the singular effort policy to be

\[
E_s(t) = a(t) - b(t)x_s(t) - \frac{pa'(t) + b'(t)c - 2pb'(t)x_s(t)}{4pb(t)x_s(t) - cb(t) - p(a(t) - \rho)}.
\]  
(15)

We emphasize that \(E_s(t)\) is a function of \(t\) only. It can be easily verified that the singular effort policy reduces to the policy developed by Clark ([1]) in case the coefficients \(a(t)\) and \(b(t)\) are constants.

Note that the functions \(x_s(t)\) and \(E_s(t)\) are defined in terms of an implicit Eq. (14). In the next lemma we establish that this implicit equation admits a unique physically meaningful solution. We also show that this solution is periodic and that the denominator term in the equation (15) can never tend to zero. This proves that the functions \(x_s(t)\) and \(E_s(t)\) are well defined.

**Lemma 4.1.** The functions \(x_s(t)\), \(E_s(t)\) are periodic of period \(T\).

**Proof.** First we show that \(x_s(t)\) is well defined. Observe that, for each fixed \(t \geq 0\), denoted by \(\bar{t}\), (14) defines a quadratic equation in \(x(\bar{t})\) with negative constant term and positive discriminant. Thus (14) admits a unique positive root for each fixed \(\bar{t}\). Hence \(x_s(t)\) is well defined for all \(t \geq 0\). Since all the coefficients of the (14) are periodic functions of period \(T\), the periodicity of \(x_s(t)\) follows. Now, to prove that \(E_s(t)\) is well defined it is sufficient to show that the term

\[
4pb(\bar{t})x_s(\bar{t}) - (b(\bar{t})c + p(a(\bar{t}) - \rho))
\]

is different from zero for each \(\bar{t} \geq 0\). Suppose, by the way of contradiction, that the above term is equal to zero for some \(\bar{t} \geq 0\). This would imply, recognizing the fact that the above expression is the derivative of (14) with respect to \(x\), that \(x_s(\bar{t})\) is a double zero of (14) at \(t = \bar{t}\), which is a contradiction due to the fact that the constant term of (14) is negative for all \(\bar{t}\). Thus \(E_s(t)\) is well defined and its periodicity follows from its definition. \(\square\)

Let \(E^*(t)\) represent the harvest policy constructed from the singular effort policy \(E_s(t)\), which is restriction of amplitude of \(E_s(t)\) to \([E_{\min}, E_{\max}]\) and let the corresponding asymptotic periodic solution of (6) with \(E(t) = E^*(t)\) be denoted by \(x^*(t)\) which is termed as optimal solution. Note that \(E^*(t)\) will in general be a combination of singular and bang–bang controls defined as follows:

\[
E^*(t) = \begin{cases} 
E_s(t) & \text{if } E_{\min} \leq E_s(t) \leq E_{\max}; \\
E_{\min} & \text{if } E_s(t) \leq E_{\min}; \\
E_{\max} & \text{if } E_s(t) \geq E_{\max}.
\end{cases}
\]  
(16)

Clearly, \(E^*(t)\) is also a periodic function of period \(T\). Now we need to reach the solution \(x^*(t)\) optimally from a given initial state \(x(0)\). In view of the propositions 3.1, 3.2 and 4.1, the solution \(x^*(t)\) is globally asymptotically stable and hence this solution can be reached by applying the bang–bang control policy initially as follows:
\[ \tilde{E}(t) = \begin{cases} E_{\text{max}} & \text{while } x(0) > x_s(0) \text{ and } x(t) > x^*(t); \\ E_{\text{min}} & \text{while } x(0) < x_s(0) \text{ and } x(t) < x^*(t). \end{cases} \tag{17} \]

If \( E^*(t) \) is equal to the singular harvest policy \( E_s(t) \) then the above bang–bang control policy reduces to the well known policy given in terms of the switching function as follows:

\[ \tilde{E}(t) = \begin{cases} E_{\text{max}} & \text{while } \sigma(t) > 0; \\ E_{\text{min}} & \text{while } \sigma(t) < 0, \end{cases} \tag{18} \]

where

\[ \sigma(t) = x(t)\{P - \mu(t)\} - c \]

is the switching function for problem 1 and the adjoint variable is described by the equation

\[ \frac{d\mu}{dt} = \rho \mu - [pE + \mu\{a(t) - 2b(t)x - E\}]. \]

Let \( \tau \) be the minimum time at which the path \( x(t) \) generated by the bang–bang control \( E(t) = \tilde{E}(t) \) reaches the solution \( x^*(t) \). The path traced by the state under this bang–bang policy in the interval \([0, \tau]\) is termed as optimal approach path. Then the optimal harvest policy is given by

\[ E_o(t) = \begin{cases} \tilde{E}(t) & \text{for } 0 \leq t \leq \tau; \\ E^*(t) & \text{for } t > \tau, \end{cases} \]

and the optimal path is given by the trajectory generated by the above control. In view of the global asymptotic stability of the solution \( x^*(t) \), it is also possible to reach \( x^*(t) \) using a suboptimal harvest policy given by

\[ E_{so}(t) = E^*(t) \text{ for } t \geq 0. \]

The path traced by the state under this suboptimal harvest policy is termed as suboptimal path. Clearly the advantage in choosing the optimal harvest policy is that the state reaches the optimal solution \( x^*(t) \) in a finite time while in the case of the said suboptimal harvest policy it reaches the optimal solution asymptotically.

4.2. Optimal harvest policy for problem 2

Since the procedure to obtain the optimal harvest policy in this case is similar to the previous one we list the important equations with explanation where ever it is necessary. We observe that the function \( f = y(z(t) - \beta(t) \ln(y)) \) is a \( C^1 \) function on the set \( y > 0 \). The current value hamiltonian for this problem is given by

\[ \mathcal{H}(y, \lambda, t) = [pE(t)y(t) - cE(t)] + \lambda \{z(t) - \beta(t) \ln(y) - E\} \]

which is linear in the control variable \( E \). Dynamics of the adjoint variable \( \lambda(t) \) for this problem is described by

\[ \frac{d\lambda}{dt} = \rho \lambda - [pE + \lambda\{z(t) - \beta(t) \ln(y) - \beta(t) - E\}]. \]
The switching function for this control problem is characterized by
\[ \zeta(t) = y(t)\{P - \lambda(t)\} - c. \]
The singular trajectory for the optimal control problem (Eq. (5)), is given by the positive solution \( y_s(t) \) of the equation
\[ \beta(t)y(\ln(y) + 1) + (\rho - \alpha(t))y - \frac{c}{\rho}(\rho + \beta(t)) = 0 \] (19)
and the corresponding singular effort is given by
\[ E_s(t) = \alpha(t) - \beta(t)\ln(y_s(t)) - \frac{c}{\rho}\beta'(t) - y_s(t)\beta'(t)\ln(y_s(t)) - \alpha'(t) - \frac{\beta'(t)}{y_s(t)[\beta(t)(\ln(y_s(t)) + 2) - \alpha(t) + \rho]} . \] (20)

The following lemma establishes that the functions \( y_s(t) \) and \( E_s(t) \) are well defined and that they are periodic of period \( T \).

**Lemma 4.2.** The functions \( y_s(t) \), \( E_s(t) \) are periodic of period \( T \).

**Proof.** To prove existence of \( y_s(t) \) for all fixed \( t \), consider the function
\[ F(y) = y\{\beta(t)(\ln(y) + 1) + (\rho - \alpha(t))\}. \]
Since
\[ \lim_{y \to 0^+} F(y) = 0 \quad \text{and} \quad \lim_{y \to +\infty} F(y) = +\infty, \]
and \( F(y) \) is continuous, there exists \( y_s > 0 \) such that
\[ F(y_s) = \frac{c}{\rho}(\rho + \beta(t)) > 0. \]
Uniqueness follows from the positive concavity of \( F \) and positivity of \( F(y_s) \). To prove that \( E_s(t) \) is well defined it is enough to show that the term
\[ y_s(t)[\beta(t)(\ln(y_s(t)) + 2) - \alpha(t) + \rho] \]
is non-zero at \( y_s \). But
\[ \beta(t)(\ln(y_s(t)) + 2) - \alpha(t) + \rho = \{\beta(t)(\ln(y_s(t)) + 1) - \alpha(t) + \rho\} + \beta(t) = \frac{F(y_s)}{y_s} + \beta(t) \]
and from (19) we conclude that
\[ \beta(t)(\ln(y_s(t)) + 2) - \alpha(t) + \rho = \frac{c}{\rho} \frac{(\rho + \beta(t))}{y_s} + \beta(t) > 0. \]
The periodicity of \( E_s(t) \) follows easily since \( y_s(t) \) is periodic of period \( T \). \( \square \)

**5. Numerical simulations**

In this section we compute numerical solutions for the optimal harvesting problems considered in the previous section. We assume the following explicit periodic forms for the coefficient functions.
We also assume the values of the parameters and initial stock levels to be

\[ a = 5, \quad c = 1, \quad \rho = 0.01, \quad E_{\text{min}} = 0, \quad E_{\text{max}} = 1, \quad x(0) = 1, \quad y(0) = 0.8. \]

To solve the problem 1, Eq. (14) is solved for positive \( x(t) \) at each \( t > 0 \) to obtain the associated singular solution \( x_s(t) \). This function \( x_s(t) \) is used in Eq. (15) to obtain the corresponding singular effort \( E_s(t) \). Similarly, Eqs. (19) and (20) are used for the problem 2 to compute corresponding singular solution \( y_s(t) \) and singular effort \( E_s(t) \). We observe that for both the problems the periodic optimal singular effort \( E_s(t) \) belongs to the interval \([0,1] = [E_{\text{min}}, E_{\text{max}}]\). Thus we have singular effort policies to be the corresponding suboptimal effort policies, i.e., \( E_s(t) = E^*(t) \) and the singular solutions of the problems to be the corresponding singular optimal solutions, i.e., \( x_s(t), y_s(t) \). Now, these optimal singular solutions are used as reference to compute the bang–bang control policy \( E^*(t) \), along the lines discussed in Section 4, to reach the corresponding optimal singular solution from a given initial position. It is to be noted that, if the singular effort \( E_s(t) \) does not belong to the set \([E_{\text{min}}, E_{\text{max}}]\) then the Eq. (16) has to be used to compute \( E^*(t) \) and this function has to be plugged into the Eq. (15) or (20), depending on the problem, to obtain \( x^*(t) \) or \( y^*(t) \), respectively.

The results of the numerical simulations for the problem 1 (problem 2) are shown in Figs. 1–5 (6–10). Fig. 1 (6) represents the singular optimal solution, and the suboptimal path. Fig. 2 (7) represents the singular effort which is also the suboptimal effort policy. Observe that the suboptimal path approaches the corresponding singular optimal solution asymptotically under the suboptimal effort policy. The optimal effort policy and the corresponding optimal path are presented in Figs. 3 and 4 (8 and 9), respectively. It is clear from these figures that the optimal effort is discontinuous and jumps from \( E_{\text{max}}, (E_{\text{min}}) \) to the singular effort policy when the optimal approach path reaches the optimal singular solution. The net revenues to the harvester for implementing the optimal and suboptimal policies are shown in Fig. 5 (10). These figures illustrate the superiority of the optimal solution in terms of net revenues to the harvester. In problem 2, where the initial level of the stock lies below the initial value of the singular optimal solution (Figs. 6–10), the long term gains to harvester by choosing the optimal to the suboptimal effort policy is evident from Fig. 10. Note that the net revenue functions corresponding to the optimal and suboptimal policies cross each other at a critical time \( t \approx 3.1 \) and for all future times beyond this critical time the net revenue due to the optimal harvest policy is higher. This implies that the short term loss to the harvester is more than compensated by the long term gains when he/she chooses to wait until the state reaches the singular optimal solution.

6. Conclusions

In this work we studied the bio-economics associated with two classical growth models, namely, the logistic and Gompertz equations, in a seasonally changing environment. Seasonality is incorporated into the models by assuming periodic growth rates and periodic carrying capacities. We
derived optimal harvest policies and optimal solutions for both the models. The optimal harvest policy is meant to maximize the net revenue to the harvester on the infinite horizon. We proved that the considered models admit a unique globally asymptotically stable periodic orbit each under periodic harvesting. In case of the logistic equation with periodic coefficients the existence of a unique globally asymptotically stable periodic orbit is already established [13]. Here we offer a

Fig. 1. Suboptimal path approaching the periodic singular optimal solution under the singular suboptimal effort policy for the logistic equation.

Fig. 2. Periodic singular suboptimal effort policy for the logistic equation.
A simpler proof for the same. This method is extended to the Gompertz equation to derive conclusions on the asymptotic behavior of its solutions.

Next, we employ the well known Pontryagin’s maximum principle to derive the optimal effort policies for both the growth models. These policies are derived as an application to a general optimal harvesting problem considered on the infinite horizon where the control variable, i.e., effort,
enters linearly into the hamiltonian. This procedure can be easily extended to diverse situations where price, cost are also seasonally dependant along with stock and effort. This method can also be applied to the cases where the growth dynamics of the resource is perturbed seasonally varying forcing functions provided the dynamic constraint in the optimal harvesting problem admits a un-

Fig. 5. Net revenue to the harvester for implementing optimal and suboptimal effort policies in case of logistic equation.

Fig. 6. Suboptimal path approaching the periodic singular optimal solution under the singular suboptimal effort policy for the Gompertz equation.
ique globally stable periodic solution for a reasonable periodic harvesting effort, i.e., whenever the harvesting function belongs to the control set $[E_{\min}, E_{\max}]$.

The investigation revealed that optimal harvesting policies and the optimal paths are periodic after a finite time and can be easily computed. For the logistic case we have explicit formulas for the optimal policy and for the Gompertz equation, the optimal policy can be computed after solv-
ing a simple transcendental equation. We proved that the suboptimal harvest policies and the optimal solutions are well defined and periodic. We also observe that the optimal solutions are globally asymptotically stable under the respective suboptimal harvesting policies. The explicit knowledge of the optimal solution (derived using the results developed in this work) allows the

Fig. 9. Optimal path (which is a combination of the optimal approach path and the singular optimal solution) for the Gompertz equation.

Fig. 10. Net revenue to the harvester for implementing optimal and suboptimal effort policies in case of Gompertz equation.
harvester to design appropriate optimal harvest policy to easily reach the optimal solution from any initial stock value. Implementation of the optimal and suboptimal harvest policies, using numerical simulation, illustrates the superiority of the optimal harvesting strategy. In particular we highlight the case where refraining from harvesting until the stock reaches the optimal solution pays the harvester in the long run.

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References