Itô versus Stratonovich calculus in random population growth

Carlos A. Braumann *

Department of Mathematics, Universidade de Évora, Rua Romão Ramalho 59, PT-7000-671 Évora, Portugal

Received 31 March 2004; accepted 23 September 2004
Available online 6 October 2005

Abstract

The context is the general stochastic differential equation (SDE) model \(dN/dt = N(g(N) + \sigma \xi(t))\) for population growth in a randomly fluctuating environment. Here, \(N = N(t)\) is the population size at time \(t\), \(g(N)\) is the ‘average’ per capita growth rate (we work with a general almost arbitrary function \(g\)), and \(\sigma \xi(t)\) is the effect of environmental fluctuations (\(\sigma > 0\), \(\xi(t)\) standard white noise). There are two main stochastic calculus used to interpret the SDE, Itô calculus and Stratonovich calculus. They yield different solutions and even qualitatively different predictions (on extinction, for example). So, there is a controversy on which calculus one should use.

We will resolve the controversy and show that the real issue is merely semantic. It is due to the informal interpretation of \(g(x)\) as being an (unspecified) ‘average’ per capita growth rate (when population size is \(x\)). The implicit assumption usually made in the literature is that the ‘average’ growth rate is the same for both calculi, when indeed this rate should be defined in terms of the observed process. We prove that, when using Itô calculus, \(g(N)\) is indeed the arithmetic average growth rate \(\overline{R_a}(x)\) and, when using Stratonovich calculus, \(g(N)\) is indeed the geometric average growth rate \(\overline{R_g}(x)\). Writing the solutions of the SDE in terms of a well-defined average, \(\overline{R_a}(x)\) or \(\overline{R_g}(x)\), instead of an undefined ‘average’ \(g(x)\), we prove that the two calculi yield exactly the same solution. The apparent difference was due to the semantic confusion of taking the informal term ‘average growth rate’ as meaning the same average.

© 2005 Elsevier Inc. All rights reserved.

* Tel.: +351 266745364; fax: +351 266745393.
E-mail address: braumann@uevora.pt

0025-5564/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
Keywords: Population growth; Itô calculus; Stratonovich calculus; Random environments; Stochastic differential equations

1. Introduction

Let \( N = N(t) \) be the size (number of individuals, biomass, density) at time \( t \geq 0 \) of a population of animals, bacteria or cells (in an organism or an organ). We assume that the population lives in an environment subjected to random fluctuations. We shall refer to \( \frac{dN}{dt} \) by total growth rate and to the per capita growth rate \( \frac{1}{N} \frac{dN}{dt} \) simply by growth rate. We can model the dynamics by assuming that the growth rate \( \frac{1}{N} \frac{dN}{dt} \) is the sum of an ‘average’ growth rate \( g(N) \) (a deterministic and usually density-dependent component) and perturbations caused by the random environmental fluctuations. Assuming a small correlation time for such perturbations, we can approximate them by a white noise \( \sigma(t) \), where \( \sigma > 0 \) is the (per capita) noise intensity and \( \epsilon(t) \) is a standard continuous-time white noise (a generalized stationary non-correlated Gaussian process, formally the generalized function derivative of the non-differentiable standard Wiener process \( w(t) \)). As usual, we have abbreviated \( N(t) = N(t, \omega) \) and \( w(t) = w(t, \omega) \), not writing explicitly the dependence of these stochastic processes on chance \( \omega \) over \( X \), where \( X \) is the set of all possible random environmental scenarios endowed with a probability space structure \( (\Omega, \mathcal{A}, P) \). By a scenario \( \omega \) we mean a specific possible arrangement of what environmental conditions a population might experience from time \( t = 0 \) to time \( t = \infty \). Each fixed scenario \( \omega \) defines a trajectory (function only of time). Usually, we only observe one trajectory, the one corresponding to the environmental scenario ‘assigned by chance’ to the particular population we are observing.

We obtain the stochastic differential equation (SDE) model

\[
\frac{1}{N(t)} \frac{dN(t)}{dt} = g(N(t)) + \sigma \epsilon(t), \quad N(0) = N_0 > 0, \quad (1)
\]

and assume the initial population size \( N_0 \) to be known. If \( \sigma = 0 \) instead of being positive, we would have a deterministic ordinary differential equation (ODE) model.

We can also consider a generalization, where we allow the noise intensity \( \sigma \) to be density-dependent \( (\sigma(N)) \) instead of being constant.

Models of this sort have been proposed in the literature, and their properties studied for specific growth rate functions [like the logistic \( g(N) = r(1 - N/K) \), the Gompertz \( g(N) = r \ln(K/N) \), and several others] and specific noise intensity functions [usually \( \sigma(N) \equiv \sigma \), but sometimes also \( \sigma(N) = c g(N) \)]. [1] was the pioneer work but many others have dealt with the subject in the following years like, for instance, [2–7] and references therein. But one could not be sure whether the properties obtained for these specific growth rate function models were valid only for the specific cases or were indeed general properties of populations growing in a random environment. This is a real problem since, with the available evidence, the correctness of any specific growth function is far from being established. So, it would be nice to obtain similar results for a general model with an arbitrary growth function satisfying only mild regularity assumptions and some general assumptions dictated by biological considerations. This was done in [8]. In [9] these results were further generalized to arbitrary noise intensities functions \( \sigma(N) \) satisfying some reasonable assumptions. Probabilities of passing by low or by high thresholds and their first passage times
are relevant for extinction studies, environmental protection and pest control; these issues can be seen, for a certain class of models, in [10] and references therein.

Similar models with harvesting $\frac{1}{N(t)} \frac{dN(t)}{dt} = g(N(t)) - h(N(t)) + \sigma(N) \varphi(t)$ have been studied for specific growth rate, noise intensity, and harvesting rate ($h(N)$) functions. See, for example, [11–16]. In [17] the main results were generalized to arbitrary growth rate and harvesting rate functions, assuming constant noise intensity. In [18] they were further generalized to arbitrary noise intensities. For the question of optimal harvesting studied as a stochastic control problem, one can see for example [19–21].

The issue of parameter estimation, testing, and prediction based on discrete observations on the single trajectory usually available can be seen in [6,14,22,23].

Here, we will resolve, with an interesting interpretation, the long standing controversy of which stochastic calculus, Itoâ or Stratonovich, is more appropriate in dealing with the general model (1) when modeling population growth in a random environment. The further generalization to an even more general SDE model where the noise intensity, instead of being constant, is allowed to be a density-dependent function $\sigma(N)$, will be treated in a forthcoming paper. Let me anticipate, though, that the conclusions for this density-dependent noise intensity case are very similar to the ones obtained here, requiring only slight adaptations in the interpretation, although the mathematical derivations become quite more technical.

The controversy comes from the fact that the two stochastic calculi do not yield the same solution to the SDEs. Let us look at the origin of this problem. The SDE model (1) can be written in the form

$$dN(t) = G(N(t)) \, dt + \Sigma(N(t)) \, dw(t), \quad N(0) = N_0 > 0,$$

where

$$G(N) = g(N)N$$

is the ‘average’ total growth rate when population size is $N$ and

$$\Sigma(N) = \sigma N$$

is the total noise intensity (intensity of the effect of random environmental fluctuations on the total growth rate $dN/dt$) when population size is $N$. The SDE can also be written as an equivalent stochastic integral equation

$$N(t) = N_0 + \int_0^t G(N(s)) \, ds + \int_0^t \Sigma(N(s)) \, dw(s). \quad (2)$$

Assuming adequate regularity conditions, the first integral can, for each trajectory (i.e., for fixed $\omega$), be defined as a Riemann integral. However, the second integral cannot be defined as a Riemann–Stieltjes integral since the Wiener process $w(t)$ is a function almost surely of unbounded variation. There are many possible definitions of this stochastic integral, but the ones commonly used in the literature are the Itoâ and the Stratonovich integrals. The Itoâ integral has nice probabilistic properties (it has zero expectation and a convenient expression for its variance, besides being a martingale as a function of $t$) but does not satisfy ordinary rules of calculus. A new stochastic calculus (Itoâ calculus) had to be developed, in which the chain rule of differentiation is different from the usual one. The Stratonovich integral does not have so nice probabilistic properties
but the corresponding calculus, the Stratonovich calculus, satisfies the usual rules of calculus (including the usual chain rule of differentiation). In Section 2, we briefly describe the differences between the two calculi.

According to which calculus one uses (i.e., according to the type of integral one uses in interpreting the equivalent stochastic integral equation (2)), we have different solutions to the SDE (1). One wonders which results are the correct ones and there is controversy in the literature over which calculus is indeed appropriate to model population growth in a random environment. In Section 3 we refer to that controversy and illustrate the consequences for the population behavior and for some ecological theories of the differences between the two calculi. For instance, even in the simple example of density-independent growth, where we have a constant ‘average’ growth rate \( (g(N) \equiv r) \), the consequences are not only quantitative. One important qualitative difference is that, using Stratonovich calculus, extinction would (similarly to the deterministic model) occur with probability one if the ‘average’ growth rate \( r \) is <0, but, with Itô calculus, one can have extinction with probability one even for positive values of the ‘average’ growth rate \( r \) (if \( r < \sigma^2/2 \)). We also speak about the main recommendations one can find in the literature on which circumstances to use Itô calculus and on which circumstances to use Stratonovich calculus. Such recommendations are, however, not helpful in the majority of the cases.

That lead us to try to shed some light on the issue in [24]. However, the parametric approach taken there was quite limited. Although appropriate to resolve the problem in the density-independent growth model and although it could be used asymptotically as \( t \to \infty \) for a certain class of density-dependent models, it was not appropriate for most density-dependent models. Here, we take an approach that, although founded on the same basic idea, is completely different. This new approach works for all density-dependent models and completely and exactly (not just asymptotically) elucidates the difference between the two calculi and interprets it in terms of the dynamics of the population. It also exactly solves in all cases the problem of which calculus to use and how to use it.

In Section 4, we illustrate the basic idea behind our new method using the simple example of density-independent growth. We show that, while one may be using the same letter \( r \) in both calculi to refer to the ‘average’ growth rate (as it is common in the literature), the interpretation of ‘\( r \)’ in terms of population dynamics is completely different under the two calculi. Indeed, we will show that, under Itô calculus, \( r \) indeed means the arithmetic average growth rate and, under Stratonovich calculus, \( r \) indeed means the geometric average growth rate (which is the right average to use for comparison purposes with the deterministic case, since population growth has a multiplicative type of dynamics). Taking into account the difference between the two averages, the two calculi give exactly the same results after all. In particular, both calculi give extinction with probability one when the geometric average growth rate is negative. The apparent difference between the two calculi was merely semantic and due solely to the wrong implicit assumption that the letter ‘\( r \)’ meant the same thing under the two calculi, the so-called ‘average’ growth rate. That wrong implicit assumption prevails in the literature, which also implicitly assumes that the term ‘average’ (the quotes are ours), without specification of what type of average, is unequivocal and therefore that the same ‘average’ is used in both calculi. This is not true and it is the source of the whole controversy.

In Section 5, we extend these results for the general model (1), again showing that the ‘average’ growth rate \( g(N) \) indeed means an arithmetic or a geometric average according to whether one
uses Itô or Stratonovich calculus. Again, taking into account the difference between the two averages, the two calculi give exactly the same results. In particular, under certain biologically reasonable conditions on \( g(N) \), extinction will occur with probability one for negative geometric average growth rates at low population densities (more exactly, at the zero limit population density). The forthcoming paper that further generalizes to the case of density-dependent noise intensities \( \sigma(N) \) reaches similar conclusions, but the interpretation of \( g(N) \) under the Stratonovich calculus is slightly different: it is a modified geometric average growth rate.

Section 6 presents the main conclusions, particularly exact recommendations on which calculus to use and how to use it. It turns out that it does not matter which calculus one uses, as long as one uses for the expression of the ‘average’ growth rate \( g(N) \) the expression that correctly describes the appropriate average for that calculus. Namely, for the general model (1), one should use the arithmetic average if one uses Itô calculus and the geometric average if one uses Stratonovich calculus. If one adopts such recommendation, the two calculi give exactly the same results.

2. Itô and Stratonovich calculi

As already mentioned, the SDE (1) is equivalent to the stochastic integral equation (2), but the second integral in (2) cannot be defined, for a fixed trajectory, as the classical Riemann–Stieltjes integral. In fact, for sequences of decompositions \( 0 = t_{0,n} \leq t_{1,n} \leq \cdots \leq t_{n,n} \) (\( n = 1, 2, \ldots \)) with diameters converging to zero, the Riemann–Stieltjes sums

\[
\sum_{i=1}^{n} \Sigma(N(\tau_{i,n}))(w(t_{i,n}) - w(t_{i-1,n}))
\]

have different mean square (m.s.) limits depending on the choice of the intermediate points \( \tau_{i,n} \in [t_{i-1,n}, t_{i,n}] \). Among the many choices, the non-anticipative choice \( \tau_{i,n} = t_{i-1,n} \) defines the Itô integral (we are assuming sufficient regularity). Its nice probabilistic properties come from the fact that the non-anticipative choice turns \( \Sigma(N(\tau_{i,n})) \) independent from the increments \( w(t_{i,n}) - w(t_{i-1,n}) \) of the Wiener process. Such an increment measures the accumulated environmental fluctuations in the time interval \( [t_{i-1,n}, t_{i,n}] \); each unit of such accumulated fluctuations affects growth rate by a factor \( \Sigma(N(\tau_{i,n})) = \Sigma(N(t_{i-1,n})) \) and Itô calculus says that such factor depends on conditions at the beginning of the interval and cannot ‘guess’ future conditions.

Itô calculus does not satisfy ordinary rules. In particular, it satisfies a different chain rule of differentiation. Namely, if \( Y(t) = f(t, N(t)) \), with \( f(t, x) \) of class \( C^{1,2} \) in \( (t, x) \), we would get

\[
(I) \quad \frac{dY}{dt} = \left( \frac{\partial f(t, N)}{\partial t} + \frac{\partial f(t, N)}{\partial x} G(N) + \frac{1}{2} \frac{\partial^2 f(t, N)}{\partial x^2} \Sigma^2(N) \right) dt + \frac{\partial f(t, N)}{\partial x} \Sigma(N) dw
\]

instead of the usual rule (applicable to Stratonovich calculus)

\[
(S) \quad \frac{dY}{dt} = \left( \frac{\partial f(t, N)}{\partial t} + \frac{\partial f(t, N)}{\partial x} G(N) \right) dt + \frac{\partial f(t, N)}{\partial x} \Sigma(N) dw.
\]

We have used ‘(I)’ or ‘(S)’ to distinguish between the Itô and the Stratonovich calculi, as we will do from now on. We have also further abbreviated \( N(t), Y(t), w(t) \) [which were already abbreviations
of \(N(t, \omega), Y(t, \omega), w(t, \omega)\) to \(N, Y, w\), and we will do that from now on. In comparison with (4), there is an extra term in (3). It has to do with the fact that second order differentials \((dw(t))^2\) are of the order of \(dt\) (because \(E[(\Delta w(t))^2] = \Delta t\)) and cannot be discarded. This is a consequence of the irregularity of the Wiener process. In fact, if \(w(t)\) was a smooth process with a derivative (it is not), we would have \((dw(t))^2 = (w'(t)dt)^2\), which would be of order \((dt)^2\) and would be discarded.

A stochastic calculus satisfying the usual rules of calculus (including the usual chain rule of differentiation (4)) uses the Stratonovich integral, which, under adequate regularity conditions, is the m.s. limit of the Riemann–Stieltjes sums

\[
\sum_{i=1}^{n} \left( \frac{\Sigma(N(t_{i-1,n})) + \Sigma(N(t_{i,n}))}{2} \right) (w(t_{i,n}) - w(t_{i-1,n})).
\]

These sums average the non-anticipative and the totally anticipative sums and, in so doing, they smooth out the irregularities caused by the Wiener process. This integral ‘guesses’ a bit into the future and it does not have the nice probabilistic properties of the Itô integral.

The Itô and Stratonovich calculi are the ones commonly used in the literature. For more details on the stochastic calculi, see, for instance, [25,26]. These books and, just to cite a few, also [27–29], can be seen for more details on stochastic differential equations.

According to which calculus one uses (i.e., according to the type of integral one uses in interpreting the equivalent stochastic integral equation (2)), we have different solutions to the SDE (1). As a matter of fact, as can be seen for instance in [25–29], under appropriate regularity conditions, the solution of the Itô SDE

\[
(I) \quad dN(t) = G(N(t)) \, dt + \Sigma(N(t)) \, dw(t)
\]

is a homogeneous diffusion process with \textit{drift coefficient}

\[
a_i(N) = G(N) = Ng(N)
\]

and \textit{diffusion coefficient}

\[
b(N) = \Sigma^2(N) = N^2 \sigma^2.
\]

On the other hand, the solution of the Stratonovich SDE

\[
(S) \quad dN(t) = G(N(t)) \, dt + \Sigma(N(t)) \, dw(t)
\]

is a homogeneous diffusion process with the same \textit{diffusion coefficient} but with a different \textit{drift coefficient}

\[
a_s(N) = G(N) + \frac{1}{4} \frac{db(N)}{dN} = N \left( g(N) + \frac{\sigma^2}{2} \right).
\]

Therefore, under appropriate regularity conditions, the Stratonovich equation

\[
(S) \quad dN = G(N) \, dt + \Sigma(N) \, dw
\]

is equivalent (i.e., has the same solution) to the Itô equation

\[
(I) \quad dN = \left( G(N) + \frac{1}{4} \frac{\Sigma^2(N)}{dN} \right) \, dt + \Sigma(N) \, dw
\]

and an obvious reverse conversion formula can also be obtained.
3. The controversy

To illustrate the consequences of the difference between the two calculi, let us consider the mal-
thusian (density-independent) growth in a random environment. It corresponds to a constant ‘average’ growth rate \( g(N) \equiv r \). To make the distinction between the two calculi more transparent we will use \( r_i \) for the Itô calculus and \( r_s \) for the Stratonovich calculus. There is no need to distinguish between the noise intensities because they play an identical role in both calculi.

Let us start by studying the density-independent Stratonovich calculus model

\[
(S) \quad dN = r_s N \, dt + \sigma N \, dw(t), \quad N(0) = N_0.
\]

(10)

It is sometimes convenient to work in logarithmic scale by making the change of variable

\[
Y(t) = \ln N(t).
\]

This is legitimate if \( N \) is dimensionless (for instance the number of individuals in the population). If \( N \) is a biomass or a population density, it would be illegitimate to do this; we would have to work instead with \( \ln(N(t)/A) \), where \( A \) would be equal to one measurement unit, but the conclusions would be the same. Notice that \( Y(t) = f(N(t)) \) with \( f(x) = \ln x \) and that \( Y(0) = Y_0 := \ln N_0 \). Since Stratonovich calculus obeys an ordinary chain rule of differentiation, we have

\[
dY = \frac{dY}{dx} \, dx = \frac{dY}{dx} \, dt + \frac{dY}{dx} \, dw(t).
\]

or the equivalent integral equation

\[
Y(t) = Y_0 + \int_0^t r_s \, dt + \int_0^t \sigma \, dw(t), \quad Y(0) = Y_0
\]

from which one immediately obtains the solution

\[
Y(t) = Y_0 + r_s t + \sigma w(t).
\]

(11)

Note that the integrand is constant; in this case, the Itô and Stratonovich integrals coincide and we have for both calculi \( \int_0^t \sigma \, dw(t) = \sigma(\omega(t) - w(0)) = \sigma w(t) \) (since \( w(0) = 0 \)). In fact, for both calculi, the integral is, in this case, the m.s. limit of the sums \( \sum_{i=1}^n \sigma(\omega(t_i, n) - \omega(t_{i-1}, n)) = \sigma(\omega(t) - w(0)) \).

Since \( w(t) \sim \mathcal{N}(0, t) \) (meaning that \( w(t) \) has a normal distribution with mean zero and variance \( t \)), we conclude that

\[
Y(t) \sim \mathcal{N}(Y_0 + r_s t, \sigma^2 t).
\]

We can see that, in log scale \( Y(t) = \ln N(t) \), the process has an average behavior identical to the deterministic solution (the solution for the case \( \sigma = 0 \)). In terms of the original scale, the solution is

\[
N(t) = N_0 \exp(r_s t + \sigma w(t))
\]

(12)

and we see that \( N(t) \) has a lognormal distribution; however, the expected value of \( N(t) \) is

\[
N_0 \exp\left( r_s t + \frac{\sigma^2}{2} t \right),
\]

different from the deterministic solution \( N_0 \exp(r_s t) \).

From (11), one obtains (with probability one) the asymptotic result \( Y(t) \sim r_s t \) as \( t \to +\infty \) (because \( w(t)/t \to 0 \) as \( t \to +\infty \)). Therefore, with probability one, as \( t \to +\infty \), \( N(t) \to +\infty \) (growth without bound) or \( N(t) \to 0 \) (extinction) according to whether the ‘average’ growth rate \( r_s \) is positive or negative. This behavior is similar to the deterministic case.
Let us look now to the Itô calculus model

\[(I) \quad dN = r_i N \, dt + \sigma N \, dw(t), \quad N(0) = N_0.\]  

We obtain \(dY = \left(\frac{df(N)}{dx} r_i N + \frac{1}{2} \frac{df(N)}{dx^2} (\sigma N)^2\right) dt + \left(\frac{df(N)}{dx} N\sigma\right) dw\) by the Itô chain rule (3). Since \(df(x)/dx = 1/x\) and \(d^2f(x)/dx^2 = -1/x^2\), we see that \(Y(t)\) satisfies the SDE

\[(I) \quad dY = \left(r_i - \frac{\sigma^2}{2}\right) \, dt + \sigma \, dw(t) \quad Y(0) = Y_0\]

or the equivalent integral equation

\[Y(t) = Y_0 + \left(r_i - \frac{\sigma^2}{2}\right) t + \sigma w(t).\]  

We conclude that

\[Y(t) \sim \mathcal{N}(Y_0 + (r_i - \sigma^2/2) t, \sigma^2 t).\]

One can see that, in log scale \(Y(t) = \ln N(t)\), the process has an average behavior different from the deterministic solution. In terms of the original scale, the solution is

\[N(t) = N_0 \exp\left((r_i - \sigma^2/2)t + \sigma w(t)\right)\]  

and we see that \(N(t)\) has a lognormal distribution; the expected value of \(N(t)\) is \(N_0 \exp(r_i t)\) and coincides with the deterministic solution.

From (14), one obtains (with probability one) the asymptotic result \(Y(t) \sim (r_i - \sigma^2/2)t\) as \(t \to +\infty\). Therefore, with probability one, as \(t \to +\infty\), \(N(t) \to +\infty\) (growth without bound) or \(N(t) \to 0\) (extinction) according to whether the ‘average’ growth rate \(r_i\) is larger than \(\sigma^2/2\) or smaller than \(\sigma^2/2\). We have a qualitatively quite different behavior from the Stratonovich calculus and from the deterministic case. Indeed, under Itô calculus, we can have extinction with probability one even for positive values of the ‘average’ growth rate. This is probably one of the reasons Itô calculus is quite popular, since the authors can send the striking message that we may get qualitatively wrong conclusions if we ignore environmental random variations and use a simple deterministic model. For that purpose, Stratonovich calculus looks quite dull. Of course, there are many other good reasons why one should use a stochastic model in certain circumstances but they may not be so striking.

Somewhat similar conclusions hold for the more general density-dependent model (1) with \(g(N)\) smooth enough and strictly decreasing. Under Stratonovich calculus (let us use \(g_d(N)\)), we will have, as can be seen in [8], that population size converges as \(t \to \infty\) to a fixed random variable having a probability density function (called stationary density) if \(g_d(0^+)\) is positive. On the contrary, if the ‘average’ growth rate at low population densities \(g_d(0^+)\) is negative, extinction will occur with probability one. If one, however, uses Itô calculus (let us use \(g_i(N)\)), we will have a stationary density or extinction according to whether the ‘average’ growth rate at low population densities \(g_i(0^+)\) is larger or smaller than \(\sigma^2/2\). Again, the same type of qualitative difference as in the density-independent case. These results (see [9]) can even be extended, with slight adaptations, to the more general model where we also allow density-dependent noise intensities \(\sigma(N)\) (satisfying appropriate technical conditions).
The results of [8,9] can be extended for populations under harvesting with a per capita harvesting rate (also called harvesting effort) \( h(N) \). With Stratonovich calculus (see [17,18]), we will get stationary density or extinction (with probability one) according to whether the ‘average’ growth rate at low population densities is higher or lower than the harvesting rate. With Itô calculus, we can get extinction (with probability one) even when the ‘average’ growth rate at low population densities is higher than the harvesting rate.

These dramatic differences show up in other places and were the source of some controversy. In [30] (see also [2,31]), May and MacArthur developed the theory of niche limiting similarity for a community of competing species in a random environment based on a system of Lotka–Volterra type Itô SDEs. Characterizing the niche by one dimensional variable, they showed that the system would not persist (in a certain sense) unless the (‘average’) niche overlap (niche similarity) was below some threshold. However, as pointed out in [5,32], the theory fails if one uses Stratonovich calculus; actually, but that is beside the point we want to make, [32] points out some flaws in May and MacArthur’s derivation. Ref. [24] calls the attention, in a context pertinent to the niche limiting similarity theory, to the meaning of the model parameters in terms of population size.

Let us fix our attention in a specific example, say the density-independent model referred to above. One may ask whether \( r \) (of the Stratonovich calculus model (10)) and \( r \) (of the Itô calculus model (13)) represent the same ‘average’ growth rate. Or may they represent different types of ‘average’? In next section we will see that this is indeed the case.

The literature does not present these questions and implicitly assumes that the letter ‘\( r \)’ it uses in both cases means the same thing under the two calculi, the so-called ‘average’ growth rate. By the same token, it is also implicitly assumed that the term ‘average’, without specification of what type of average, is unequivocal and therefore that the same ‘average’ is used in both calculi. As we will see, this is not true and it is the source of the whole controversy.

The controversy went on with recommendations on what is the appropriate calculus to model population growth in random environments. The point is that a SDE is always an approximation. Two main situations are usually considered for the way nature behaves:

(a) The phenomenon occurs in discrete time and it would be better described by a stochastic difference equation (SAE) of the form

\[
\Delta N = G(N(t))\Delta t + \Sigma(N(t))\tilde{\epsilon}(t)\Delta t \quad (t = 0, \Delta t, 2\Delta t, \ldots, n\Delta t; \quad N(0) = N_0),
\]

where \( \tilde{\epsilon}(t)(t = 0, \Delta t, 2\Delta t, \ldots, n\Delta t) \) is a discrete-time white noise (just a sequence of independent standard Gaussian random variables, therefore a proper stochastic process). One, however, prefers to use a SDE as a more tractable approximation. One can see (for instance in [25]) that, for \( G \) and \( \Sigma \) sufficiently regular, the solution of the SAE converges in the supremum norm (with probability one) to the solution of the Itô SDE (5) when \( \Delta t \to 0 \). In this case, the Itô calculus for the SDE would be the advisable approximation.

(b) The phenomenon occurs in continuous time but the noise is not white. In fact, no one really believes that the effects of environmental fluctuations on growth rate can be described by a continuous-time white noise \( \sigma(t) \), a generalized stochastic process with independent values at two different time instants, no matter how close these two instants are. This is possible only because white noise is a generalized process; no proper stochastic process has that property. More likely, natural continuous-time noises \( \tilde{\epsilon}(t) \) are colored, showing some correlation
between close-by instants and the white noise is only a convenient mathematical approximation when the correlation time is small. While the integral of white noise is the non-differentiable Wiener process, which has independent increments, the integral of the real noise is likely to be a ‘near’-Wiener process, i.e. a smoother differentiable process $\tilde{w}(t)$ with increments showing a slight dependence for neighbor time intervals. In that case, the stochastic integrals $\int_0^t \Sigma(N(t)) \, \tilde{w}(t) \, dt$ can, for each trajectory (fixed $\omega$), be computed as Riemann–Stieljes integrals, which satisfy ordinary calculus rules. Suppose now that the process $\tilde{w}(t)$ converges in m.s. to $w(t)$ in an appropriate fashion and that $G$ and $\Sigma$ are sufficiently regular. One would expect that the solution of the ‘smooth’ SDE

$$dN(t) = G(N(t)) \, dt + \Sigma(N) \, d\tilde{w}(t) \quad N(0) = N_0$$

in a finite time interval $[0, T]$ would converge in the supremum norm (with probability one) to the solution of the Stratonovich SDE (8). Indeed (see, for instance, [25]), this result holds even for polygonal approximations $\tilde{w}(t)$ of $w(t)$ (assuming the corresponding decompositions of $[0, T]$ have diameters converging to zero). In this case, the Stratonovich calculus for the SDE would be the advisable approximation.

From these considerations, it would seem advisable to use the Itô calculus when the population growth process intrinsically occurs in discrete time (non-overlapping generations) and to use the Stratonovich calculus when the process intrinsically occurs in continuous time (overlapping generations) (see [3,33,34]). That, of course, requires the validity of the appropriate conditions for the above limit results to hold. If the conditions are not so appropriate, the limit results may not hold (see [35,36]).

But the problem goes beyond that. The problem is that the above recommendations are for the most part worthless and people can and do argue which of the recommendations applies to a particular situation. Actually, in the context of deterministic models, there is a similar debate whether to use difference or differential equations to model population growth. In fact, it is not all that clear whether population growth intrinsically occurs in discrete or continuous time. Even in populations with clear and regular non-overlapping generations (like many bird and insect populations), although births occur in discrete evenly spaced seasons, the fact is that a season, however short, is not an instant. So, births spread over the season. But, even if we consider such spread in births sufficiently small to be ignored, what about deaths? They have no seasons and occur at any time. On the other hand, in populations with overlapping generations, births and deaths are discrete events, although they can potentially occur at any instant in a continuous time set.

It would therefore seem ill-advised to base predictions, particularly when related to relevant issues like extinction and ecological theories, on the researcher’s feelings on whether population growth intrinsically occurs in discrete or continuous time. Of course, the same type of issues arise in several other applications of SDEs. The alternative is the pragmatic view, now quite popular in the SDE literature: use the calculus that seems to work better for the particular situation at hand.

Our point, as we shall show, is that it does not matter which calculus one uses, it does not matter whether or not population growth intrinsically occurs in discrete or continuous time. What matters is to get rid of the semantic confusion of assuming that $g(N)$ means the same ‘average’ growth rate under the two calculi. What matters is to use for the ‘average’ growth rate $g(N)$
the appropriate average for the calculus one is using. If we do so, the two calculi yield exactly the same results.

4. The resolution for density-independent growth

Let us go back to the density-independent growth model (10) or (13), respectively for Stratonovich and for Itô calculus, considered for illustration purposes in the previous section. The model corresponds to a constant \( g(N) \). Although, in the literature on the Itô-Stratonovich controversy, the same letter \( r \) is used in both models to represent an implicitly common ‘average’ growth rate, we did use \( r_s \) and \( r_i \) to make the distinction between the two calculi more transparent. We also asked the question whether or not they indeed represent the same ‘average’ growth rate. Let us now answer the question.

For that, we need to clarify what growth rate and ‘average’ growth rate mean in terms of the observed population dynamics \( N(t) \), which is the only objective information on the population. In fact, we do not know \( r_s \) or \( r_i \) (or even \( r \)); we have to estimate them from observations on the population size trajectory \( N(t) \) (or equivalent information on births and deaths over time).

Let us examine first the deterministic model, the ordinary differential equation (ODE)

\[
dN = r_d N \, dt \quad N(0) = N_0.
\]

Suppose that, at a given instant \( t \), the observed population size is \( N(t) = x \) and we want to know what is the (per capita) growth rate \( R(x) \) at that instant \( t \). Even for the more general density-dependent models, it obviously does not depend on \( t \) but only on the value \( x \) the population size has at that time; this is due to the fact that these models are autonomous equations (the function \( G(N) \) does not depend on time). In this density-independent case, \( R(x) \) does not depend on population size \( x \) either, but we will still use the notation \( R(x) \) in this particular case, since the dependence on \( x \) will be present in the more general density-dependence situation to be dealt with in the next section. Obviously, \( R(x) \) is the total growth rate \( dN(t)/dt \) divided by the population size \( N(t) = x \), i.e.

\[
R(x) = \frac{1}{x} \left( \frac{dN}{dt} \right)_{N=x} = \frac{1}{x} \lim_{\Delta t \to 0} \frac{N(t + \Delta t) - x}{\Delta t}.
\]

Looking at (16), we have the obvious conclusion that

\[
R(x) = r_d.
\]

Of course, we could have gone through the unnecessary steps of obtaining the solution \( N(t) = N_0 \exp(r_d t) \) of (16), noticing that \( N(t + \Delta t) = N(t)\exp(r_d \Delta t) = x \exp(r_d \Delta t) \) and replacing in (17) to get to the same conclusion.

Let us now consider the Itô SDE (13) or the Stratonovich SDE (10). At the instant \( t + \Delta t \), the population size \( N(t + \Delta t) \) is now a random variable. So, the limit value \( R(x) \) of (17) (which in this stochastic setting will also not depend on \( t \)) is random. So, we need to take some type of average. One idea would be to first compute the limit in (17) and take some average afterwards. This does not work. In fact, due to the non-differentiability of the Wiener process in the ordinary sense, the limit in (17) is a generalized stochastic process (involving white noise) and does not exist in the
ordinary sense and so the averages are ill-defined. So, we will take the average of random quantities first and the limit afterwards. But we have to be precise on what type of average we are using.

The first idea is to use the arithmetic average, that is, since we are dealing with random variables, the usual expected value. Of course, since we know that at time \( t \) the population size is \( x \), we should take the expectation conditioned on that knowledge. We will denote by \( \mathbf{E}_{t,x} \) such conditional expectations, i.e. \( \mathbf{E}_{t,x}[\cdots] = \mathbf{E}[\cdots|N(t) = x] \). We can now define the arithmetic average growth rate as

\[
R_a(x) := \frac{1}{x} \lim_{\Delta t \to 0} \frac{\mathbf{E}_{t,x}[N(t + \Delta t)] - x}{\Delta t}.
\]

There are many other types of averages. For instance, the geometric average is obtained by transforming the quantities to be averaged to log scale, then take an ordinary arithmetic average (expected value), and finally revert to the initial scale by inverting the logarithm. So, we can define the geometric average growth rate as

\[
R_g(x) := \frac{1}{x} \lim_{\Delta t \to 0} \exp\left(\frac{\mathbf{E}_{t,x}[\ln N(t + \Delta t)] - x}{\Delta t}\right).
\]

Let us compute these two averages for the Stratonovich SDE model (10). Since we know that \( N(t) = x \), we get from (11) and for \( \Delta t > 0 \),

\[
Y(t + \Delta t) = \ln x + r_s \Delta t + \sigma(w(t + \Delta t) - w(t)).
\]

Therefore, conditioned on \( N(t) = x \), \( Y(t + \Delta t) \) is \( \mathcal{N}(\ln x + r_s \Delta t, \sigma^2 \Delta t) \) and so its conditional expectation is \( \mathbf{E}_{t,x}[Y(t + \Delta t)] = \mathbf{E}_{t,x}[\ln N(t + \Delta t)] = \ln x + r_s \Delta t \). Replacing into (19), we obtain

\[
R_g(x) = \frac{1}{x} \lim_{\Delta t \to 0} \exp\left(\frac{\ln x + r_s \Delta t - x}{\Delta t}\right) = r_s.
\]

Of course, conditioned on \( N(t) = x \), we see that \( N(t + \Delta t) = \exp(Y(t + \Delta t)) \) is lognormal with parameters \( \ln x + r_s \Delta t \) and \( \sigma^2 \Delta t \); so, its conditional expectation is \( \mathbf{E}_{t,x}[N(t + \Delta t)] = \exp(\ln x + r_s \Delta t + \sigma^2 \Delta t/2) = x \exp((r_s + \sigma^2/2)\Delta t) \). Replacing into (18) we obtain

\[
R_a(x) = \frac{1}{x} \lim_{\Delta t \to 0} \frac{x \exp((r_s + \sigma^2/2)\Delta t) - x}{\Delta t} = r_s + \frac{\sigma^2}{2}.
\]

The conclusion of (20) is that, \textit{when using Stratonovich calculus}, the so-called ‘average’ growth rate \( r_s \) is not any unspecified average, it is the geometric average growth rate.

Let us now compute the two averages for the Itô SDE model (13). Since we know that \( N(t) = x \), we get from (14) and for \( \Delta t > 0 \),

\[
Y(t + \Delta t) = \ln x + \left(r_i - \frac{\sigma^2}{2}\right)\Delta t + \sigma(w(t + \Delta t) - w(t)).
\]

Therefore, conditioned on \( N(t) = x \), we see that \( Y(t + \Delta t) \) is normally distributed with mean \( \ln x + \left(r_i - \frac{\sigma^2}{2}\right)\Delta t \) and variance \( \sigma^2 \Delta t \) and so its conditional expectation is \( \mathbf{E}_{t,x}[Y(t + \Delta t)] = \mathbf{E}_{t,x}[\ln N(t + \Delta t)] = \ln x + \left(r_i - \frac{\sigma^2}{2}\right)\Delta t \). Replacing into (19), we obtain
\[ R_g(x) = \frac{1}{x} \lim_{\Delta t \to 0} \frac{\exp \left( \ln x + \left( r_i - \frac{\sigma^2}{2} \right) \Delta t \right) - x}{\Delta t} \equiv r_i - \frac{\sigma^2}{2}. \]  

(22)

Of course, conditioned on \( N(t) = x \), we see that \( N(t + \Delta t) = \exp(Y(t + \Delta t)) \) is lognormal with parameters \( \ln x + (r_i - \frac{\sigma^2}{2}) \Delta t \) and \( \sigma^2 \Delta t \); so, its conditional expectation is \( \mathbb{E}_{t,x}[N(t + \Delta t)] = \exp(\ln x + (r_i - \frac{\sigma^2}{2}) \Delta t + \sigma^2 \Delta t/2) = x \exp(r_i \Delta t). \) Replacing into (18) we obtain

\[ R_a(x) = \frac{1}{x} \lim_{\Delta t \to 0} \frac{x \exp(r_i \Delta t) - x}{\Delta t} \equiv r_i. \]  

(23)

The conclusion of (23) is that, when using Itô calculus, the so-called ‘average’ growth rate \( r_i \) is not any unspecified average, it is the arithmetic average growth rate.

Let us now replace the unspecified ‘average’ growth rate \( r \) (\( r_s \) or \( r_i \)) by the specified average it really represents. Only in this way the results acquire real meaning. Since \( r_s \) is indeed the geometric average growth rate \( R_g \) (let us drop the ‘(x)’ in this density-independent case), if we replace in (12), we conclude that the solution of the Stratonovich SDE density-independent growth model is

\[ N(t) = N_0 \exp(R_g t + \sigma w(t)). \]

Since \( r_i \) is indeed the arithmetic average growth rate \( R_a \), if we replace in (15), we conclude that the solution of the Itô SDE density-independent growth model is

\[ N(t) = N_0 \exp \left( (R_a - \frac{\sigma^2}{2}) t + \sigma w(t) \right) \]

or, since \( R_a - \frac{\sigma^2}{2} = R_g \),

\[ N(t) = N_0 \exp(R_g t + \sigma w(t)). \]

We conclude that the two calculi yield exactly the same solution in terms of a specific average growth rate. It does not matter which average we choose as long as it is clearly specified (and not some obscure unspecified ‘average’ growth rate as commonly done in the literature); we have written the common solution in terms of the geometric average growth rate, but we can as well write it in terms of the arithmetic average. Looking back at extinction, we conclude that both calculi predict, with probability one, population growth without bound or extinction according to whether the geometric average growth rate is positive or negative.

So, we can use either calculus indifferently as long as we are careful to use for \( r \) the appropriate average for that calculus:

(a) the arithmetic average growth rate value when using Itô calculus;
(b) the geometric average growth rate value when using Stratonovich calculus.

5. The resolution for density-dependent growth

We will now consider the more general model (1) with density-dependent ‘average’ growth rate \( g(N) \). To make more transparent the distinction between the Itô and the Stratonovich calculus we will use \( g_s(N) \) and \( g_s(N) \), respectively. There is no need to distinguish between the noise intensities because they play an identical role in both calculi.
Let us first consider the Itô calculus and write the SDE in the standard form:

\[
(I) \quad dN(t) = g_i(N(t))N(t) \, dt + \sigma N(t) \, dw(t), \quad N(0) = N_0 > 0.
\]

We will assume that

(A) \( g_i(\cdot) : (0, +\infty) \mapsto (-\infty, +\infty) \) is a continuously differentiable function.

(B) The limit \( g_i(0^+) := \lim_{N \to 0^+} g_i(N) \) exists; it may be finite, \(+\infty\), or \(-\infty\); in any case the limit \( G_i(0^+) := \lim_{N \to 0^+} g_i(N)N \) must also exist and \( G_i(0^+) = 0 \) (no spontaneous generation).

Usually, the limit \( g_i(0^+) \) exists and is finite, in which case we get automatically \( G_i(0^+) = 0 \). However there are some models proposed in the literature, though somewhat unrealistic, for which the limit \( g_i(0^+) \) is infinite, but they still verify the no spontaneous generation assumption. A well-known such model example is the Gompertz model \( (g_i(x) = r \ln(K/x)) \), which approaches an infinite per capita growth rate when population size approaches zero.

With assumptions (A) and (B), there is a unique strong solution up to an explosion time (see, for instance, [25]). Assume further that:

(C) The boundaries \( N = 0 \) and \( N = +\infty \) of (24) are unattainable.

Then, with probability one, the explosion time is \(+\infty\) (no explosion occurs) and we conclude that a unique solution \( N(t) \) of (24) exists for all \( t \geq 0 \) and has values in the interval \((0, +\infty)\). One can see in [27,29] technical conditions for general Itô SDEs to have unattainable boundaries.

The solution is a homogeneous diffusion process with (see (6) and (7)) drift coefficient

\[
a_i(x) = G_i(x) = g_i(x) \quad (25)
\]

and diffusion coefficient

\[
b(x) = \Sigma^2(x) = \sigma^2 x^2 \quad (26)
\]

(they do not depend on \( t \) because our SDE is autonomous, i.e., \( G \) and \( \Sigma \) do not depend on \( t \)). These coefficients are the moments (if they exist)

\[
a_i(x) := \lim_{\Delta t \to 0} \frac{\mathbb{E}_{t,x}[(N(t + \Delta t) - x)]}{\Delta t} \quad (27)
\]

and

\[
b(x) := \lim_{\Delta t \to 0} \frac{\mathbb{E}_{t,x}[(N(t + \Delta t) - x)^2]}{\Delta t} \quad (28)
\]

If the moments (27) and (28) do not exist, the coefficients are equal to the truncated moments

\[
a_i(x) := \lim_{\Delta t \to 0} \frac{\mathbb{E}_{t,x}[(N(t + \Delta t) - x)I_M]}{\Delta t}
\]

and

\[
b(x) := \lim_{\Delta t \to 0} \frac{\mathbb{E}_{t,x}[(N(t + \Delta t) - x)^2 I_M]}{\Delta t}.
\]
The truncation, when necessary, is carried by the indicator function \( I_M \), which is equal to 1 when \( |N(t + \Delta t) - x| \leq M \) (\( M \) is a fixed arbitrarily large positive number) and is equal to 0 otherwise. Assume now that:

(D) The moments (27) and (28) for the solution of the SDE (24) exist.

With these assumptions, we get (27) and therefore (see its definition (18)) the arithmetic average growth rate exists and is given by

\[
R_a(x) := \lim_{x \to t} \frac{E_{t,x}[N(t + \Delta t) - x]}{\Delta t} = \frac{a_i(x)}{x} = g_i(x). \tag{29}
\]

To compute the geometric average growth rate, let us go to log scale \( Y(t) = \ln N(t) \). Using Itô chain rule (3), we obtain the SDE

\[
(I) \quad dY = \left( g_i(e^y) - \frac{\sigma^2}{2} \right) dt + \sigma dw(t), \quad Y(0) = Y_0 = \ln N_0. \tag{30}
\]

Since the boundaries \( N = 0 \) and \( N = +\infty \) of the solution \( N(t) \) of (24) are unattainable, \( Y(t) \) will stay in \(( -\infty, +\infty)\) (will not explode). In terms of \( Y = \ln N \) and putting \( y = \ln x \), the solution is a diffusion process with drift coefficient

\[
\bar{a}_i(y) = g_i(e^y) - \frac{\sigma^2}{2}
\]

and diffusion coefficient

\[
\bar{b}(y) = \sigma^2.
\]

These coefficients are the moments

\[
\bar{a}_i(y) := \lim_{\Delta t \to 0} \frac{E_{t,x}[Y(t + \Delta t) - y]}{\Delta t} \tag{31}
\]

and

\[
\bar{b}(y) := \lim_{\Delta t \to 0} \frac{E_{t,x}[(Y(t + \Delta t) - y)^2]}{\Delta t}, \tag{32}
\]

or, if the moments do not exist, they are equal to the corresponding truncated moments. Assume now that:

(E) The moments (31) and (32) for the solution of the SDE (30) exist.

With these assumptions, we get

\[
g_i(e^y) - \frac{\sigma^2}{2} = \lim_{\Delta t \to 0} \frac{E_{t,x}[Y(t + \Delta t) - y]}{\Delta t}. \tag{33}
\]

From here we get, when \( \Delta t \to 0 \),

\[
E_{t,x}[\ln N(t + \Delta t)] = E_{t,x}[Y(t + \Delta t)] = y + (g_i(e^y) - \frac{\sigma^2}{2}) \Delta t + o(\Delta t)
\]
and so
\[ \exp \left( E_{t,x} \left[ \ln N(t + \Delta t) \right] \right) = x \exp \left( (g_i(x) - \sigma^2/2) \Delta t + o(\Delta t) \right) = x \left( 1 + (g_i(x) - \sigma^2/2) \Delta t + o(\Delta t) \right). \]

Therefore (see the definition (19)) the geometric average growth rate exists and is given by
\[ R_g(x) := \frac{1}{x} \lim_{\Delta t \to 0} \frac{\exp(E_{t,x} \left[ \ln N(t + \Delta t) \right]) - x}{\Delta t} = g_i(x) - \sigma^2/2. \]

(34)

In view of (33), we can also use the following equivalent alternative and suggestive expression for the geometric average growth rate:
\[ R_g(x) = \lim_{\Delta t \to 0} \frac{E_{t,x} \left[ \ln N(t + \Delta t) \right] - \ln x}{\Delta t}. \]

(35)

Let us summarize the conclusions we have reached:

**Theorem 1.** Consider the SDE
\[ dN(t) = g_i(N(t))N(t) \, dt + \sigma N(t) \, dw(t), \quad N(0) = N_0 > 0 \]
satisfying assumptions (A), (B), and (C), which ensure existence of a unique solution \( N(t) \) with values in the interval \((0, +\infty)\) defined for all \( t \geq 0 \). Then:

(a) If assumption (D) holds, then \( g_i(x) \) is the arithmetic average growth rate
\[ R_a(x) := \frac{1}{x} \lim_{\Delta t \to 0} \frac{E_{t,x} \left[ N(t + \Delta t) \right] - x}{\Delta t}. \]

(b) If assumption (E) holds, then \( g_i(x) - \sigma^2/2 \) is the geometric average growth rate
\[ R_g(x) := \frac{1}{x} \lim_{\Delta t \to 0} \frac{\exp(E_{t,x} \left[ \ln N(t + \Delta t) \right]) - x}{\Delta t} = \lim_{\Delta t \to 0} \frac{E_{t,x} \left[ \ln N(t + \Delta t) \right] - \ln x}{\Delta t}. \]

Observation. If assumption (D) [respectively assumption (E)] does not hold, the arithmetic [respectively geometric] average growth rate will not exist but we can replace it by a truncated version (which exists for arbitrarily large truncation thresholds) and this truncated average is equal to \( g_i(x) \) [respectively \( g_i(x) - \sigma^2/2 \)].

The important thing to retain is that, under appropriate conditions, when using Itô calculus, the so-called ‘average’ growth rate \( g_i(x) \) is not any unspecified average, it is the arithmetic average growth rate.

Of course, to reach such conclusion we have made the assumptions (A), (B), (C), and (D). Assumptions (A) and (B) are technical but they are biologically reasonable; in particular assumption (B) implies no spontaneous generation. Are assumptions (C) and (D) also reasonable assumptions for population growth models? We will examine that issue right away, but, before doing so, notice that, if assumption (E) also holds,
\[ R_g(x) = R_a(x) - \sigma^2/2. \]
We try now to establish conditions on the growth rate that ensure the validity of conditions (C), (D), and (E). Consider the following assumptions:

(F) There are $A \in (-\infty, +\infty)$ and $B \in (0, +\infty)$ such that $g_i(x) \leq A$ for $x > B$.

(G) If it happens that $g_i(0^+) = +\infty$, then we must have $g_i(x) \leq C(1 + 1/x^2)$ (with $C$ some positive finite constant) for $x$ in some neighborhood $(0, \delta)$ (with $\delta > 0$) of zero.

(H) If it happens that $g_i(0^+) = -\infty$, then we must have $g_i(x) \geq D \ln x$ (with $D$ some positive finite constant) for $x$ in some neighborhood $(0, \delta)$ (with $\delta > 0$) of zero.

Assumption (F) is quite natural and says that, for large populations, the per capita growth $g_i(x)$ is bounded above (in real populations is even negative, but here we do not forbid it to be positive). The condition is so liberal that even density-independent growth ($g_i(x) \equiv r$) satisfies it. Notice that the technical assumptions (G) and (H) are required only in the unusual situation where the limit $g_i(0^+)$ is infinite; in fact, when that limit is finite, assumptions (G) and (H) are automatically satisfied. Even the Gompertz model satisfies these assumptions.

**Theorem 2.** Consider the SDE

$$dN(t) = g_i(N(t))N(t)dt + \sigma N(t)dw(t), \quad N(0) = N_0 > 0$$

with $g_i$ satisfying assumptions (A) and (B).

If assumptions (F), (G), and (H) hold, then assumptions (C), (D), and (E) become automatically satisfied and so, by Theorem 1:

(a) The SDE has a unique solution $N(t)$ with values in the interval $(0, +\infty)$ defined for all $t \geq 0$.

(b) $g_i(x)$ is the arithmetic average growth rate $R_a(x)$ defined by (18).

(c) $g_i(x) - \sigma^2/2$ is the geometric average growth rate $R_g(x)$ defined by (19) or by the equivalent expression (35).

**Proof.** See Appendix A. □

Let us now consider the *Stratonovich calculus* and write the SDE in the standard form:

$$dN(t) = g_s(N(t))N(t)dt + \sigma N(t)dw(t), \quad N(0) = N_0 > 0.$$  \hspace{1cm} (36)

The solution of the Stratonovich SDE (36) is a diffusion process with diffusion coefficient

$$b(x) = \sigma^2 x^2,$$  \hspace{1cm} (37)

identical to the diffusion coefficient (26) of the Itô SDE (24). However, if we would mimic most literature and not make the distinction between $g_i(x)$ and $g_s(x)$, the drift coefficient

$$a_s(x) = (g_s(x) + \sigma^2/2)x,$$  \hspace{1cm} (38)

of the Stratonovich SDE (36) would apparently be different from the drift coefficient $a_i(x) = g_i(x)x$ (see (25)) of the Itô SDE (24). Of course, the solutions would also appear different. We would also conclude (with appropriate assumptions), as we have seen in Section 3, that extinction would occur with probability one when the ‘average’ growth rate $g(0^+)$ at low population densities is negative or smaller than $\sigma^2/2$, according to whether we use Stratonovich or Itô calculus.
To analyze the Stratonovich SDE (36), we are going to convert it by the method referred to at the end of Section 2, thus obtaining the equivalent Itô SDE

\[(I) \quad \mathrm{d}N(t) = N(t)(g_s(N(t)) + \sigma^2/2) \, \mathrm{d}t + N(t)\sigma \, \mathrm{d}w(t), \quad N(0) = N_0 > 0.\]  

Although this equation looks different from (36), it has the same solution. This trick will save us a lot of work, since we can now work in the realm of Itô calculus and take advantage of the results already obtained in this section. Notice that the Itô SDE (39) is of the same type as the Itô SDE (24) but with \(g_i(x)\) replaced by \(g_s(x) + \sigma^2/2\). So, all considerations made above for (24) can be translated to our present situation just by replacing \(g_i(x)\) by \(g_s(x) + \sigma^2/2\).

Assumptions (A) to (D) translate to:

\[(A') \quad g_s(\cdot) : (0, +\infty) \mapsto (-\infty, +\infty) \text{ is a continuously differentiable function.}\]

\[(B') \quad \text{The limit } g_s(0^+) := \lim_{N \to 0} g_s(N) \text{ exists; it may be finite, } +\infty, \text{ or } -\infty; \text{ in any case the limit } G_s(0^+) := \lim_{N \to 0} g_s(N)N \text{ must also exist and } G_s(0^+) = 0.\]

\[(C') \quad \text{The boundaries } N = 0 \text{ and } N = +\infty \text{ of the SDE (36) (or of the equivalent SDE (39)) are unattainable.}\]

\[(D') \quad \text{The non-truncated moments } a_s(x) := \lim_{\Delta t \to 0} \frac{E_x[g_s(N(t+\Delta t)-x)]}{\Delta t} \text{ and } b(x) := \lim_{\Delta t \to 0} \frac{E_x[(N(t+\Delta t)-x)^2]}{\Delta t} \text{ for the solution of the SDE (36) (or of the equivalent SDE (39)) exist.}\]

To compute the geometric average growth rate, let us go to log scale \(Y(t) = \ln N(t)\). We can start from the Stratonovich SDE (36) and use the ordinary chain rule of differentiation (Stratonovich calculus satisfies the ordinary rules). We obtain the SDE

\[(S) \text{ and } (I) \quad \mathrm{d}Y = g_s(e^Y) \, \mathrm{d}t + \sigma \, \mathrm{d}w(t), \quad Y(0) = Y_0 = \ln N_0,\]  

which can be indifferently interpreted as an Itô or Stratonovich equation. In fact, since the stochastic term has a constant coefficient, the additive correction term of the conversion method at the end of Section 2 is now zero and the two calculi coincide. We would have reached the same result if we had started, not from the Stratonovich SDE (36), but from the equivalent Itô SDE (39) and then use the Itô chain rule of differentiation.

Putting \(y = \ln x\), assumption (E) now translates to:

\[(E') \quad \text{The non-truncated moments } \tilde{a}_s(x) := \lim_{\Delta t \to 0} \frac{E_x[y(N(t+\Delta t)-y)]}{\Delta t} \text{ and } \tilde{b}(x) := \lim_{\Delta t \to 0} \frac{E_x[(y(N(t+\Delta t)-y)^2]}{\Delta t} \text{ for the solution of the SDE (40) exist.}\]

As for the assumptions (F), (G), and (H), they now translate to (constants in inequalities may be different after translation, but we keep the same letters for convenience):

\[(F') \quad \text{There are } A \in (-\infty, +\infty) \text{ and } B \in (0, +\infty) \text{ such that } g_s(x) \leq A \text{ for } x > B.\]

\[(G') \quad \text{If it happens that } g_s(0^+) = +\infty, \text{ then we must have } g_s(x) \leq C(1 + 1/x^2) \text{ (with } C \text{ some positive finite constant) for } x \text{ in some neighborhood } (0, \delta) \text{ (with } \delta > 0) \text{ of zero.}\]

\[(H') \quad \text{If it happens that } g_s(0^+) = -\infty, \text{ then we must have } g_s(x) \geq D \ln x \text{ (with } D \text{ some positive finite constant) for } x \text{ in some neighborhood } (0, \delta) \text{ (with } \delta > 0) \text{ of zero.}\]
The same translation mechanism translates (29) and (34) to
\[ R_a(x) := \lim_{\Delta t \to 0} \frac{E_{t,x}[N(t + \Delta t)] - x}{\Delta t} = g_s(x) + \sigma^2/2 \]
and
\[ R_g(x) := \lim_{\Delta t \to 0} \frac{\exp(E_{t,x}[\ln N(t + \Delta t)]) - x}{\Delta t} = g_s(x). \]

These are, respectively, the arithmetic average growth rate and the geometric average growth rate for the solution of the Stratonovich SDE (36).

The important thing to retain is that, under appropriate conditions, when using Stratonovich calculus, the so-called ‘average’ growth rate \( g_s(x) \) is not any unspecified average, it is the geometric average growth rate. The geometric average is the appropriate average for comparison with the deterministic model since the growth process is of a multiplicative nature; that is why the Stratonovich results on extinction looked very similar to the deterministic results. Also, the conditions for the deterministic model to have population size converging to a positive stable equilibrium look similar to conditions of existence of a stationary density (a kind of stochastic equilibrium) in the Stratonovich stochastic model.

Again, for the Stratonovich SDE (36), we also have (no surprise: that is a property of these averages, not of the calculus used)
\[ R_g(x) = R_a(x) - \sigma^2/2. \]

Theorems 1 and 2 translate now to:

**Theorem 1’. Consider the SDE**
\[ (S) \quad dN(t) = g_s(N(t))N(t)\,dt + \sigma N(t)\,d\omega(t), \quad N(0) = N_0 > 0 \]
satisfying assumptions (A’), (B’), and (C’), which ensure existence of a unique solution \( N(t) \) with values in the interval \((0, +\infty)\) defined for all \( t \geq 0 \). Then:

(a) If assumption (D’) holds, then \( g_s(x) + \sigma^2/2 \) is the arithmetic average growth rate
\[ R_a(x) := \lim_{\Delta t \to 0} \frac{E_{t,x}[N(t + \Delta t)] - x}{\Delta t}. \]

(b) If assumption (E’) holds, then \( g_s(x) \) is the geometric average growth rate
\[ R_g(x) := \lim_{\Delta t \to 0} \frac{\exp(E_{t,x}[\ln N(t + \Delta t)]) - x}{\Delta t} = \lim_{\Delta t \to 0} \frac{E_{t,x}[\ln N(t + \Delta t)] - \ln x}{\Delta t}. \]

**Observation.** If assumption (D’) [respectively assumption (E’)] does not hold, the arithmetic [respectively geometric] average growth rate will not exist but we can replace it by a truncated version (which exists for arbitrarily large truncation thresholds) and this truncated average is equal to \( g_s(x) + \sigma^2/2 \) [respectively \( g_s(x) \)].
Theorem 2’. Consider the SDE

\[ dN(t) = g_s(N(t))N(t)dt + \sigma N(t)dw(t), \quad N(0) = N_0 > 0 \]

with \( g_s \) satisfying assumptions \( (A') \) and \( (B') \).

If assumptions \( (F') \), \( (G') \), and \( (H') \) hold, then assumptions \( (C') \), \( (D') \), and \( (E') \) become automatically satisfied and so, by Theorem 1’:

(a) The SDE has a unique solution \( N(t) \) with values in the interval \( (0, +\infty) \) defined for all \( t \geq 0 \).

(b) \( g_s(x) + \sigma^2/2 \) is the arithmetic average growth rate \( R_a(x) \) defined by (18).

(c) \( g_s(x) \) is the geometric average growth rate \( R_g(x) \) defined by (19) or by the equivalent expression (35).

Let us now replace the unspecified ‘average’ growth rate \( g(x) \) (\( g_s(x) \) or \( g_i(x) \)) by the specified average it really represents. Only in this way the results acquire real meaning. Since \( g_s(x) \) is indeed the geometric average growth rate \( R_g(x) \), replacing in \( a_s(x) = x(g_s(x) + \sigma^2/2) \) (see (38)), we obtain \( a_s(x) = x(R_g(x) + \sigma^2/2) \). Taking into account that \( R_g(x) = R_a(x) - \sigma^2/2 \), we get

\[ a_s(x) = xR_a(x). \]

Since \( g_s(x) \) is indeed the arithmetic average growth rate \( R_a(x) \), replacing in \( a_i(x) = xg_i(x) \) (see (25)), we obtain

\[ a_i(x) = xR_a(x). \]

We have seen that the diffusion coefficient \( b(x) \) is the same for both calculi.

One concludes that the two calculi yield exactly the same diffusion process (same drift and diffusion coefficients) and so the Itô SDE (24) and the Stratonovich SDE (36) have exactly the same solution in terms of a specific average growth rate. It does not matter which average we choose as long as it is clearly specified (and not some obscure unspecified ‘average’ growth rate as commonly done in the literature); we have written the common drift and diffusion coefficients of the common solution in terms of the arithmetic average growth rate, but we can as well write them in terms of the geometric average. Looking back at Section 3 with smooth strictly decreasing \( g(N) \), we conclude that both calculi predict, with probability one, existence of a stationary density or extinction according to whether the geometric average growth rate is positive or negative.

Again, we see that one can use either calculus indifferently as long as he/she is careful to use for \( g(N) \) the appropriate average for that calculus:

(a) the arithmetic average growth rate value when using Itô calculus;

(b) the geometric average growth rate value when using Stratonovich calculus.

6. Conclusions

In the context of density-dependent (and also density-independent) SDE population growth models in a randomly fluctuating environment (as well as in other contexts), there was a long standing controversy of whether it is more appropriate to use Itô or Stratonovich calculus. The two calculi lead to qualitatively different results on extinction and existence of a stationary density
(a kind of stochastic equilibrium). For instance, for \( g \) strictly decreasing, while under Stratonovich calculus extinction occurs with probability one when the ‘average’ (per capita) growth rate \( g(0^+) \) (the ‘average’ growth rate at low population sizes) is negative, in Itô calculus extinction with probability one also occurs for a range of positive values of the ‘average’ growth rate \( g(0^+) \). In harvesting models, conditions on extinction or existence of a stationary density also depended on the calculus used. Another example is the May and MacArthur’s theory of niche limiting similarity, derived under Itô calculus, which collapses when Stratonovich calculus is used. Since both calculi are approximate models, the main recommendation (based on certain limit theorems) was to use Itô or Stratonovich according to whether population growth intrinsically occurs in discrete or continuous time. That recommendation is practically worthless because population growth is a complicated phenomenon that has discrete-time and continuous-time components.

We concluded that the whole controversy was based on a semantic confusion. In fact, most literature implicitly assumes that the ‘average’ growth rate function \( g(N) \) that shows up in the Itô and Stratonovich SDEs means the same (not specified) ‘average’ for both calculi. We have clarified what ‘average’ growth rate \( g(N) \) really means (of course, in terms of the real thing: the dynamics of the population).

It turns out, as we have proved, that, when using Itô calculus, \( g(N) \) refers to the arithmetic average growth rate, while, when using Stratonovich calculus, it is the geometric average growth rate.

So, if, instead of using in the Itô and Stratonovich SDE a meaningless letter \( g \) assumed to mean an unspecified ‘average’ growth rate function \( g(N) \), we use the average that \( g \) stands for, the semantic confusion would disappear. We should write the arithmetic average growth rate \( R_a(N) \) instead of \( g(N) \) when using Itô calculus and should write the geometric average growth rate \( R_g(N) \) instead of \( g(N) \) when using Stratonovich calculus. If we do so, it turns out, as we have proved, that the two calculi have exactly the same solution and, of course, have exactly the same quantitative and qualitative predictions. As a by-product of the resolution of the controversy, we get a nice interpretation for the real meaning of the population growth function \( g(N) \) in the two calculi.

For instance, in the example above, extinction occurs with probability one when the geometric average growth rate \( R_g(0^+) \) is negative, and that is true for both Itô and Stratonovich calculus. Of course, if that geometric average growth rate is only slightly negative, the arithmetic average is positive and that was the source of the apparent qualitatively contradiction between the two calculi. Indeed, there is no contradiction, just confusion between the geometric and the arithmetic average growth rates, taken to be the same unspecified ‘average’, which of course they are not.

So, one can use indifferently the Itô or the Stratonovich calculus, as long as he/she is careful to use for \( g(N) \) the appropriate average for that calculus:

(a) the arithmetic average growth rate value when using Itô calculus;
(b) the geometric average growth rate value when using Stratonovich calculus.

Actually, if we know one of the averages, we immediately know the other since \( R_a(N) = R_g(N) + \sigma^2/2 \), where \( \sigma \) is the (per capita) environmental noise intensity.

Both calculi can be used no matter how population is intrinsically growing, be it in discrete or in continuous time. So, we do not have to worry about that elusive question.
From now on, the issue is no longer which calculus to use. That issue is now clarified with a ‘does not matter’. Does it mean we have no more problems when applying SDE models? Not quite. There remains still a difficult issue: how to determine the correct expression for the arithmetic or the geometric growth rate (whatever one we need). Say, for instance, we are using Stratonovich calculus and therefore need to obtain the geometric average growth rate \( R_g(N) \) (the correct average interpretation for the growth function \( g(N) \) in Stratonovich calculus). The following question still remains: should \( R_g(N) \) be logistic type \( r(1 - N/K) \), Gompertz type \( r \ln(K/N) \) or some of the many other types proposed in the literature? Or some not yet proposed type? We may obtain an expression working from first principles, but different populations may have different behaviors.

We may also use parametric or non-parametric estimation approaches. However, \( R_g(x) \) is a geometric average given by expression (19); to obtain its value for a given population size \( N = x \), we would need to take the mathematical expectation in that expression. But, the mathematical expectation is an average over population trajectories (repetitions of the experiment) and we usually have only a single trajectory. Fortunately, if the process is ergodic we can replace that average by a time-average over that single trajectory (in the non-parametric approach, for example, we take averages over time instants where the population size is \( x \) or very close to \( x \)). Many models have, for a certain range of parameters, ergodic solutions. Sufficient conditions for that can be seen in [8].

In this paper, we have resolved the controversy on whether to use Itô or Stratonovich calculus for the general case, i.e., for a very general class of SDE models. Every growth rate function \( g(N) \) so far proposed in the literature (as far as I have checked) is included. Since we were dealing with environmental stochasticity (perturbations on population growth induced by environmental stochastic variability), we have assumed, as is commonly done in the literature, that the (per capita) noise intensity does not depend on population size. However, the same conclusions hold for the more general case where we allow \( r \) to become a density-dependent function \( r(N) \) satisfying very mild assumptions. The assumptions have to be consistent with environmental stochasticity (demographic stochasticity does not satisfy the assumptions). The only difference is that proofs become more technical and \( g_s(N) \) is no longer the geometric average growth rate but rather a slightly modified geometric average growth rate. This will be the object of a forthcoming paper.

**Acknowledgment**

This work was performed at CIMA-UE (Centro de Investigação em Matemática e Aplicações da Universidade de Évora), a research center financed by the Portuguese research funding agency FCT (Fundação para a Ciência e a Tecnologia) within its ‘Programa de Financiamento plurianual’ (under FEDER funding).

**Appendix A. Proof of Theorem 2**

As can be seen, for instance, in [27] (see p. 68), sufficient conditions for existence (non-explosion) and uniqueness of a solution of (24) and for the existence of the moments (27) and (28)
are: (i) $G_l(x)$ and $\Sigma(x)$ are continuous and satisfy a local Lipschitz condition; (ii) they satisfy an appropriate restriction on growth. Since $\Sigma(x) = \sigma x$ and $G_l(x) = g_l(x)x$ are continuously differentiable (use assumption (A)), (i) is immediately satisfied. The restriction on growth can be (see [27], p. 68) $G^2_l(x) \leq L(1 + |x|^2)$ and $\Sigma^2(x) \leq L(1 + |x|^2)$ for some constant $L > 0$. In our case $\Sigma(x) = \sigma x$ and so the second part is automatically satisfied but the first does not hold for many common population growth models like, for example, the logistic. However, the first part of the restriction on growth can be replaced by $xG_l(x) \leq L(1 + x^2)$ as can be seen in [27] (from the proof of the theorem in page 68 properly adapted in conjunction with theorem 4 on page 48). Remember that our state space for population size $N$ is $(0, +\infty)$.

We are now going to show that $xG_l(x) = x^2 g_l(x) \leq L(1 + x^2)$, which is equivalent to

$$g_l(x) \leq L(1 + 1/x^2) \quad \text{for some finite constant} \quad L > 0. \quad (41)$$

Since we can choose $L > A$, we see from assumption (F) that (41) holds for every $x > B$. With $0 < \delta \leq B$, it also holds for every $x \in [\delta, B]$; in fact, this is a closed interval where $g_l(x)$ is continuous (by assumption (A)), and so $g_l$ has a maximum $M$ there and we just need to put $L > M$.

It remains to check if (41) holds for $x$ in some neighborhood $(0, \delta) (\delta > 0)$ of the origin. By assumption (B), the limit $g_l(0^+) = Z$ exists. If $Z$ is finite, (41) holds because, given $\eta > 0$, there is a neighborhood $(0, \delta)$ of zero where $g_l(x) \leq Z + \eta$ (just take $L > \max(0, Z + \eta)$). It also holds when $Z = +\infty$ (just use assumption (G) and put $L \geq C$). Finally, (41) obviously holds when $Z = -\infty$ because $g_l(x)$ is negative in a small neighborhood $(0, \delta)$ of zero.

Since there is a unique solution, there is no explosion and so the boundary $N = +\infty$ is unattainable. Since we have shown that the moments (27) and (28) exist, assumption (D) holds. So, it remains to show that the boundary $N = 0$ is unattainable and that assumption (E) holds.

We will now show that the boundary $N = 0$ is unattainable. Let $\bar{g}(x) = g_l(x) - \sigma^2/2$. By assumption (B), we know that the limit $\bar{W} = \bar{g}(0^+) = g(0^+) - \sigma^2/2$ exists.

One can see, for instance in [28] or [29], that the boundary $N = 0$ is unattainable if $v(0^+) = +\infty$, where $v(x) = \int_c^x (\int_c^z m(z) dz) s(y) dy$, with $s(x) = \exp \left(- \int_c^x \frac{2g_l(\theta)}{\sigma^2} d\theta \right)$ and $m(x) = \frac{1}{s(x)H(x)}$. Of course, in our case, the drift and diffusion coefficients are $a_l(x) = g_l(x)x$ and $b(x) = \sigma^2 x^2$. The function $s(x)$ is called scale density and the function $m(x)$ is called speed density, and they are defined up to a multiplicative constant (so, constant $c$ can be arbitrarily chosen in the state space). We will choose $c > 0$ in an appropriate small neighborhood of zero.

In our case, we have

$$s(y) = \exp \left(- \int_c^y \frac{2g_l(\theta)\theta}{\sigma^2\theta^2} d\theta \right) = \exp \left(\int_c^y \frac{2\bar{g}(\theta)}{\sigma^2\theta} d\theta \right)$$

$$= \exp \left(\int_c^y \frac{2\bar{g}(\theta)}{\sigma^2\theta} d\theta + \int_y^c \frac{1}{\theta} d\theta \right) = \frac{c}{y} \exp \left(\int_y^c \frac{2\bar{g}(\theta)}{\sigma^2\theta} d\theta \right) = \frac{c}{y} \exp \left(\int_y^c H(\theta) d\theta \right),$$

with

$$H(\theta) = \frac{2\bar{g}(\theta)}{\sigma^2\theta}. $$
We can also see that \( s(y) \) and \( m(z) \) are always positive, that
\[
\begin{align*}
s(y)m(z) &= \frac{1}{\sigma^2 y^2} \exp \left( \int_y^z \frac{2g(\theta)}{\sigma^2 \theta} \, d\theta \right) = \frac{1}{2g(z) y} H(z) \exp \left( \int_y^z H(\theta) \, d\theta \right) \\
&= \frac{1}{2g(z) y} \frac{\partial}{\partial z} \exp \left( \int_y^z H(\theta) \, d\theta \right)
\end{align*}
\]
and that
\[
\begin{align*}
v(0^+) &= \int_0^c \left( \int_y^c m(z) \, dz \right) s(y) \, dy = \int_0^c \left( \int_y^c s(y) \, dy \right) m(z) \, dz,
\end{align*}
\]
where we have exchanged the order of the integration in the second integral.

We just need to show that \( v(0^+) = +\infty \).

Let us do that for the case \( W \geq 0 \) (including the case \( W = +\infty \)). In this case, we have \( g(\theta) \geq -\alpha \) with \( \alpha > 0 \) for \( \theta \) in a neighborhood \((0, \delta)\) of zero. Let \( c < \delta \). Let \( z \) run in the interval \((0, c)\), \( y \) run in the interval \((0, z)\) and \( \theta \) run in the interval \((y, z)\). Then \( g(\theta) \geq -\alpha \) and, using the first line of (42) and the last expression of (43), we obtain
\[
\begin{align*}
v(0^+) &= \int_0^c \left( \int_0^c s(y) m(z) \, dy \right) m(z) \, dz = \frac{1}{\sigma^2} \int_0^c \left( \int_0^z \frac{1}{yz} \exp \left( \int_y^z \frac{2g(\theta)}{\sigma^2 \theta} \, d\theta \right) \, dy \right) \, dz \\
&\geq \frac{1}{\sigma^2} \int_0^c \left( \int_0^z \frac{1}{yz} \exp \left( \int_y^z \frac{-2\alpha}{\sigma^2 \theta} \, d\theta \right) \, dy \right) \, dz = \frac{1}{\sigma^2} \int_0^c \left( \int_0^z \frac{1}{yz} \exp \left( \frac{-2\alpha}{\sigma^2} \, yz \right) \, dy \right) \, dz \\
&= \frac{1}{\sigma^2} \int_0^c \left( \int_0^z y^{2z/\sigma^2} \, dz \right) z^{-2z/\sigma^2} \, dz = \frac{1}{2\alpha} \int_0^c z^{-1} \, dz = +\infty.
\end{align*}
\]

Let us now show that \( v(0^+) = +\infty \) for the case \( W < 0 \) (including the case \( W = -\infty \)). If \( W = -\infty \), assume that \( \delta < 1/e \) in assumption (H) to obtain (notice that \( -\ln \theta > 1 \) \( g(\theta) \geq D \ln \theta - \sigma^2/2 \geq D \ln \theta - (\sigma^2/2)(-\ln \theta) = D^* \ln \theta \) for \( \theta \in (0, \delta) \) with \( D^* = D + \sigma^2/2 \). We also have \( g(\theta) \leq -\beta \) (with \( \beta > 0 \)) for \( \theta \) in a neighborhood \((0, \delta')\) of zero. Then, with \( \delta' = \min(\delta, \delta') > 0 \), we have
\[
D^* \ln \theta \leq g(\theta) \leq -\beta \quad \text{for} \quad \theta \in (0, \delta')
\]
for some \( D^* > 0 \) and \( \beta > 0 \). If \( W \) finite, we have \( -\alpha \leq g(\theta) \leq -\beta \) with \( \alpha > 0 \) and \( \beta > 0 \) for \( \theta \) in a neighborhood \((0, \delta)\) of zero. Since \( -\alpha \geq D^* \ln \theta \) for \( \theta \leq \exp(-\alpha/D^*) \), we still obtain (44) by choosing \( \delta' = \min(\delta, \exp(-\alpha/D^*)) \). Let \( c < \delta' \). Let \( y \) run in the interval \((0, c)\), \( z \) run in the interval \((y, c)\) and \( \theta \) run in the interval \((y, z)\). Then, using the last line of (42), the first expression of (43), and (44), we obtain
\[
\begin{align*}
v(0^+) &= \int_0^c \left( \int_y^c m(z) \, dz \right) s(y) \, dy = \int_0^c \left( \int_y^c \frac{1}{2g(z) y} \frac{\partial}{\partial z} \exp \left( \int_y^z H(\theta) \, d\theta \right) \, dz \right) \, dy \\
&\geq \int_0^c \left( \int_y^c \frac{1}{2yD^* \ln z} \frac{\partial}{\partial z} \exp \left( \int_y^z H(\theta) \, d\theta \right) \, dz \right) \, dy \\
&\geq \int_0^c \left( \int_y^c \frac{1}{2yD^* \ln y} \frac{\partial}{\partial z} \exp \left( \int_y^z H(\theta) \, d\theta \right) \, dz \right) \, dy
\end{align*}
\]
\[
\frac{1}{2D^2} \int_0^c \frac{1}{-y \ln y} \left( 1 - \exp \left( \int_y^c H(\theta) \, d\theta \right) \right) \, dy \\
= \frac{1}{2D^2} \int_0^c \frac{1}{-y \ln y} \left( 1 - \exp \left( \int_y^c \frac{2g(\theta)}{\sigma^2 \theta} \, d\theta \right) \right) \, dy \\
\geq \frac{1}{2D^2} \int_0^c \frac{1}{-y \ln y} \left( 1 - \exp \left( \int_y^c \frac{-2\beta}{\sigma^2 \theta} \, d\theta \right) \right) \, dy \\
= \frac{1}{2D^2} \int_0^c \frac{1}{-y \ln y} \left( 1 - \left( \frac{c}{y} \right)^{-2\beta/\sigma^2} \right) \, dy \\
= \frac{1}{2D^2} \int_0^c \frac{1}{-y \ln y} \, dy - \frac{1}{2D^2} \int_0^c \frac{1}{-y \ln y} \left( \frac{c}{y} \right)^{-2\beta/\sigma^2} \, dy.
\]

Since \( 0 < \int_0^c \frac{1}{-y \ln y} \left( \frac{c}{y} \right)^{-2\beta/\sigma^2} \, dy \leq \frac{c^{-2\beta/\sigma^2}}{-\ln c} \int_0^c \frac{1}{y^{2\beta/\sigma^2 - 1}} \, dy < +\infty \) and \( \int_0^c \frac{1}{-y \ln y} \, dy = -\ln(-\ln c) + \ln(-\ln 0^+) = +\infty \), we obtain \( v(0^+) = +\infty \).

This concludes the proof that the boundary \( N = 0 \) of the Itô SDE (24) is unattainable.

We will now prove that assumption (E) holds. Remember that \( Y(t) \), the solution of the Itô SDE (30), is the logarithm of \( N(t) \), which is the solution of the SDE (24). We have already shown that \( N(t) \) exists, is unique and stays in \((0, +\infty)\) (remember that \( N = 0 \) and \( N = +\infty \) were shown to be unattainable boundaries). So \( Y(t) \) exists (no \(+\infty \) or \(-\infty \) explosions) and is unique. By the results mentioned at the beginning of this appendix, if the deterministic part \( g_i(e^y) - \sigma^2/2 \) and the stochastic part \( \sigma \) of (30) are continuous and satisfy a local Lipschitz condition and appropriate restrictions on growth, then we get to the same conclusions and also insure the existence of the moments (31) and (32). The stochastic part \( \sigma \) is constant, therefore continuous and Lipschitz, and automatically satisfies the growth restriction \( \sigma^2 \leq L(1 + |x|^2) \) for the positive constant \( L = \sigma^2 \). Assumption (A) takes care of the continuity and the local Lipschitz condition for the deterministic part. We only need to check the restriction on growth for the deterministic part \( g_i(e^y) - \sigma^2/2 \). We are going to show the validity of the growth restriction
\[
y(g_i(e^y) - \sigma^2/2) \leq L(1 + y^2) \quad \text{for some finite constant} \; L > 0.
\]

We now show that (45) holds in a neighborhood of \(+\infty\). When assumption (F) holds for some values of \( A \) and \( B \), it also holds when we replace those values by larger ones; so, we can assume that \( A > \sigma^2/2 \) and \( B > 1 \). Let \( P > \max(1, \ln B) \). We will show that (45) holds for \( y > P \). Indeed, from assumption (F) we get \( g_i(e^y) - \sigma^2/2 \leq A^+ \) (with \( A^+ = A - \sigma^2/2 > 0 \)) for \( e^y < B \), i.e., for \( y > \ln B \). So, for \( y > P \) (which implies \( y > 1 \)), we get \( y(g_i(e^y) - \sigma^2/2) \leq yA^+ \leq y^2A^+ \leq A^+(1 + y^2) \). We just need to choose \( L \geq A^+ \).

It is easy to show that (45) holds for \( y \) in an arbitrary finite closed interval \([Q, P]\) since the function \( y(g_i(e^y) - \sigma^2/2) \) is continuous and therefore has a maximum \( M \) in that interval. It suffices then to choose \( L > M \) for (45) to hold for \( y \in [Q, P] \).

It remains to check if (45) holds for \( y \) in some neighborhood \((-\infty, Q)\) of \(-\infty\). We may assume \( Q < -1 \) without loss of generality. By assumption (B), the limit \( W = \lim_{y \to -\infty} (g_i(e^y) - \sigma^2/2) = g_i(0^+) - \sigma^2/2 \) exists. If \( W \) is finite, given a \( \eta > 0 \), there is a \( y \)-neighborhood \((-\infty, Q) \) of \(-\infty \) where \( |g_i(e^y) - \sigma^2/2 - W| < \eta \) and so \( |y(g_i(e^y) - \sigma^2/2)| < (|W| + \eta)|y| \leq (|W| + \eta)(1 + |y|^2) \) (because \( y < Q < -1 \) implies \( |y| > 1 \) and \( |y|^2 > |y| \)); (45) holds if we put \( L > |W| + \eta \). If \( W = +\infty \), then there
is a $y$-neighborhood $(-\infty, Q)$ (suppose $Q < -1$) of $-\infty$ where $g_i(e^y) - \sigma^2/2$ is positive and, since $y$ is negative, $y(g_i(e^y) - \sigma^2/2)$ is negative there and, consequently, satisfies (45). Finally, if $W = -\infty$, we use condition (H) with $\delta < 1/e$ and consider the $y$-neighborhood $V = (-\infty, Q)$ of $-\infty$ with $Q = \ln \delta < -1$. When $x \in (0, \delta)$, we have the required $y = \ln x \in V$ and $y(g_i(e^y) - \sigma^2/2) \leq y(Dy - \sigma^2/2) = Dy^2 + y\sigma^2/2 \leq (D + \sigma^2/2)y^2 \leq L(1 + y^2)$ if we put $L \geq (D + \sigma^2/2)$.

References